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An optimizing reduced PLSMFE formulation for non-stationary conduction–convection problems

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SUMMARY

In this paper, proper orthogonal decomposition (POD) is combined with the Petrov–Galerkin least squares mixed finite element (PLSMFE) method to derive an optimizing reduced PLSMFE formulation for the non-stationary conduction–convection problems. Error estimates between the optimizing reduced PLSMFE solutions based on POD and classical PLSMFE solutions are presented. The optimizing reduced PLSMFE formulation can circumvent the constraint of Babuška–Brezzi condition so that the combination of finite element subspaces can be chosen freely and allow optimal-order error estimates to be obtained. Numerical simulation examples have shown that the errors between the optimizing reduced PLSMFE solutions and the classical PLSMFE solutions are consistent with theoretical results. Moreover, they have also shown the feasibility and efficiency of the POD method. Copyright © 2008 John Wiley & Sons, Ltd.

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1. INTRODUCTION

Let $\Omega\subset\mathbb{R}^2$ be a bounded and connected polygonal domain. Consider the non-stationary conduction–convection problems whose coupled equations governing viscous incompressible flow

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and heat transfer for the incompressible fluid are Boussinesq approximations to the non-stationary Navier–Stokes equations.

Problem (I): Find $\mathbf{u}=(u_1,u_2)$, p and T such that for $t_N>0$,

$$
\begin{aligned}
\mathbf{u}_t-\nu\Delta\mathbf{u}+(\mathbf{u}\cdot\nabla)\mathbf{u}+\nabla p &= \mathbf{j}T, \quad (x,y,t)\in\Omega\times(0,t_N)\\
\nabla\cdot\mathbf{u} &= 0, \quad (x,y,t)\in\Omega\times(0,t_N)\\
T_t-\gamma_0^{-1}\Delta T+\mathbf{u}\cdot\nabla T &= 0, \quad (x,y,t)\in\Omega\times(0,t_N)\\
\mathbf{u}(x,y,t)=\mathbf{0}, \quad T(x,y,t) &= \varphi(x,y,t), \quad (x,y,t)\in\partial\Omega\times(0,t_N)\\
\mathbf{u}(x,y,0)=\mathbf{0}, \quad T(x,y,0) &= \psi(x,y), \quad (x,y)\in\Omega
\end{aligned}
\tag{1}
$$

where $\mathbf{u}=(u_1,u_2)$ represents the velocity vector, p the pressure, T the temperature, Re the Reynolds number, Pr the Prandtl number, $\nu=\sqrt{Pr/Re}$, $\gamma_0=\sqrt{RePr}$, $\mathbf{j}=(0,1)$ the unit vector, and $\varphi(x,y,t)$ and $\psi(x,y)$ are the given functions, while t_N is the final time. For the sake of convenience and without loss of generality, we may suppose in the following theoretical analysis that $\varphi(x,y,t)=0$.

The non-stationary conduction–convection problems (I) constitute an important system of equations in atmospheric dynamics and a dissipative nonlinear system of equations. Since this system of equations does not only contain the velocity vector field as well as the pressure field but also contain the temperature field [1, 2], finding the numerical solution of Problem (I) is a difficult task. There are at least 15 papers in a special IJNMF issue (vol. 40, issue 8) addressing this topic—comparing and discussing various numerical approaches including the Petrov–Galerkin method. In particular, we would mention document [2] of the above issue that summarizes the results from the papers dedicated to understanding the fluid dynamics of thermally driven cavity. Although mixed finite element (MFE) method is one of the important approaches for solving the non-stationary conduction–convection problems, the fully discrete system of MFE solutions for the non-stationary conduction–convection problems has many degrees of freedom and an important convergence stability condition is that the Babuška–Brezzi (BB) inequality [3, 4] holds for the combination of finite element subspaces. Thus, an important problem is how to circumvent the constraint of the BB inequality and alleviate the computational load by saving time-consuming calculations in the computational process in a way that guarantees a sufficiently accurate numerical solution.

To circumvent the constraint of the BB inequality in MFE methods for Stokes and Navier–Stokes equations, stabilized finite element methods [5–8] have been developed, motivated by the streamline diffusion methods [9, 10]. Tang and Tsang have proposed a least squares finite element method for time-dependent incompressible flows with thermal convection [11]. Some Petrov–Galerkin least squares methods for the stationary Navier–Stokes equations and the non-stationary conduction–convection problems were developed [12, 13].

Proper orthogonal decomposition (POD) is a technique for adequate approximation of fluid flow with a reduced number of degrees of freedom, i.e. with lower-dimensional models alleviating the computational load and providing CPU and memory savings. POD has been successfully used in different fields including signal analysis and pattern recognition [14, 15], fluid dynamics, and coherent structures [16–21], as well as in optimal flow control problems [22–24]. More recently, some reduced-order finite difference models and MFE formulations and error estimates for the upper tropical Pacific ocean model based on POD were presented [25–28], along with an optimizing finite difference scheme based on POD for non-stationary conduction–convection problems [29]. Kunisch and Volkwein have presented some Galerkin POD methods for parabolic

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problems [30] and a general equation in fluid dynamics [31]. The singular value decomposition approach combined with POD technique is used to treat the Burgers equation in [32] and the cavity flow problem in [33]. Patera and Rønquist have also presented a reduced basic approximation and *a posteriori* error estimation for a Boltzmann model [34]. And again, Rovas *et al.* have advanced reduced basic output bound methods for parabolic problems [35].

To the best of our knowledge, there are no published methods addressing the case where POD is used to reduce the Petrov–Galerkin least squares mixed finite element (PLSMFE) formulation for non-stationary conduction–convection problems or providing error estimates between classical PLSMFE and reduced PLSMFE solutions. In this paper, we combine PLSMFE methods with POD to deal with the non-stationary conduction–convection problems. In this manner, we ensure not only stabilization of solutions of the fully discrete PLSMFE system but also alleviate the computational load and save time-consuming calculations in the computational process while guaranteeing a sufficiently accurate numerical solution. We also derive error estimates between usual PLSMFE solutions and the solutions of optimizing reduced PLSMFE formulation based the POD technique. Then, we consider the results obtained from numerical simulations of cavity flows to show that the errors between POD solutions of optimizing reduced PLSMFE formulation and the usual PLSMFE solutions are consistent with theoretical results.

The present paper is organized as follows. In Section 2 we derive the usual PLSMFE methods for the non-stationary conduction–convection problems and generate snapshots from transient solutions computed from the equation system derived by usual PLSMFE methods. In Section 3, the optimal orthogonal bases are reconstructed from elements of the snapshots with POD and an optimizing reduced PLSMFE formulation is developed with a lower-dimensional number based on POD for the nonlinear non-stationary conduction–convection problems. In Section 4, error estimates between usual PLSMFE solutions and POD solutions of optimizing reduced PLSMFE formulation are derived. In Section 5, some numerical examples are presented illustrating that the errors between optimizing the PLSMFE approximate solutions and the usual PLSMFE solutions are consistent with previously obtained theoretical results. Section 6 provides conclusions and future tentative ideas.

2. USUAL PLSMFE APPROXIMATION FOR THE NON-STATIONARY CONDUCTION–CONVECTION PROBLEMS AND SNAPSHOTS GENERATION

The Sobolev spaces used in this context are standard [36]. Let N be a positive integer; denote the time step increment by $k=t_N/N$. The notation $t_n=kn$ and $(\mathbf{u}^n, p^n, T^n)$ denotes the semi-discrete approximation of $(\mathbf{u}(x,y,t_n), p(x,y,t_n), T(x,y,t_n))$. By introducing a finite difference approximation for time derivation of Problem (I), we obtain the following semi-discrete formulation at discrete times.

Problem (II): Find $(\mathbf{u}^n, p^n)\in X\times M$ such that for $n=1,2,\ldots,N$,

$$\begin{aligned}(\mathbf{u}^n,\mathbf{v})+ka(\mathbf{u}^n,\mathbf{v})+ka_1(\mathbf{u}^n,\mathbf{u}^n,\mathbf{v})-kb(p^n,\mathbf{v}) &= k(jT^n,\mathbf{v})+(\mathbf{u}^{n-1},\mathbf{v}) \quad \forall\mathbf{v}\in X\\ b(q,\mathbf{u}^n) &= 0 \quad \forall q\in M\\ (T^n,\phi)+kD(T^n,\phi)+ka_2(\mathbf{u}^n,T^n,\phi) &= (T^{n-1},\phi) \quad \forall\phi\in W\\ \mathbf{u}^0 &= \mathbf{0},\quad T^0=\psi(x,y),\quad (x,y)\in\Omega\end{aligned} \tag{2}$$

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where

$$X=H_0^1(\Omega)^2,\quad M=\left\{q\in L^2(\Omega),\ \int_\Omega q\,\mathrm{d}x\,\mathrm{d}y=0\right\},\quad W=H_0^1(\Omega)$$

$$a(\mathbf{u},\mathbf{v})=\nu\int_\Omega \nabla\mathbf{u}\cdot\nabla\mathbf{v}\,\mathrm{d}x\,\mathrm{d}y,\quad b(q,\mathbf{v})=\int_\Omega q\,\mathrm{div}\,\mathbf{v}\,\mathrm{d}x\,\mathrm{d}y$$

$$a_1(\mathbf{u},\mathbf{v},\mathbf{w})=\frac{1}{2}\int_\Omega\sum_{i=1}^2\sum_{j=1}^2\left[u_i\frac{\partial v_j}{\partial x_i}w_j-u_i\frac{\partial w_j}{\partial x_i}v_j\right]\mathrm{d}x\,\mathrm{d}y$$

$$a_2(\mathbf{u},T,\phi)=\frac{1}{2}\int_\Omega\sum_{i=1}^2\left[u_i\frac{\partial T}{\partial x_i}\phi-u_i\frac{\partial \phi}{\partial x_i}T\right]\mathrm{d}x\,\mathrm{d}y$$

$$D(T,\phi)=\gamma_0^{-1}\int_\Omega \nabla T\cdot\nabla\phi\,\mathrm{d}x\,\mathrm{d}y$$

Remark 1
Problem (II) uses Euler backward one step to discretize the time derivative. However, the time derivative may use other difference schemes, e.g. more exactly, central differences, forward difference, etc., but the basic approach is the same as in the present method.

Using the theory of stationary conduction–convection problems proves that Problem (II) has a unique solution and has the following error estimate [1, 35].

Theorem 2.1
If second derivatives u_{tt} and T_{tt} of the solution (u,p,T) of Problem (I) are all bounded, then

$$\|\mathbf{u}(t_n)-\mathbf{u}^n\|_0+(k\nu)^{1/2}\sum_{i=1}^n|\mathbf{u}(t_i)-\mathbf{u}^i|_1+\|T(t_n)-T^n\|_0$$
$$+(k\gamma_0^{-1})^{1/2}\sum_{i=1}^n|T(t_i)-T^i|_1+k^{1/2}\sum_{i=1}^n\|p(t_i)-p^i\|_0\leqslant Ck$$

where $(\mathbf{u}(t_n),p(t_n),T(t_n))$ is the value at $t_n=kn$ of the solution $(\mathbf{u}(t),p(t),T(t))$ of Problem (I), C is a constant depending only on $\psi(x,y)$, Reynolds number, Prandtl number and t_N but independent of k.

Throughout this paper, C indicates a positive constant that is possibly different at different occurrences and is independent of the mesh parameters h and time step increment k, but may depend on Ω, the Reynolds number, and on other parameters introduced in this paper.

In order to find the numerical solution for Problem (II), it is necessary to discretize Problem (II). We introduce an MFE approximation for the spatial variable. Let $\{\Im_h\}$ be a uniformly regular family of triangulation of $\bar{\Omega}$ [35, 36], indexed by a parameter $h=\max_{K\in\Im_h}\{h_K;h_K=\mathrm{diam}(K)\}$, i.e. there is a constant C, independent of h, such that $h\leqslant Ch_K$ $(\forall K\in\Im_h)$. We introduce the finite element subspaces $X_h\subset X$, $M_h\subset M$, and $W_h\subset W$ as follows:

$$\begin{aligned}X_h&=\{\mathbf{v}_h\in X\cap C^0(\bar{\Omega})^2;\mathbf{v}_h|_K\in P_\ell(K)^2\ \forall K\in\Im_h\}\\ M_h&=\{q_h\in M\cap C^0(\bar{\Omega});q_h|_K\in P_\kappa(K)\ \forall K\in\Im_h\}\\ W_h&=\{\phi_h\in W\cap C^0(\bar{\Omega});\phi_h|_K\in P_\iota(K)\ \forall K\in\Im_h\}\end{aligned}\tag{3}$$

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where $P_\ell(K)$ is the space of piecewise polynomials of degree ℓ on K, $\ell\geqslant 1$, $\kappa\geqslant 1$, and $\imath\geqslant 1$ are three integers.

Let $(\mathbf{u}_h^n, p_h^n, T_h^n)\in X_h\times M_h\times W_h$ be the PLSMFE approximation corresponding to $(\mathbf{u}^n, p^n, T^n)$. Then, the fully discrete PLSMFE solution for Problem (II) may be expressed as follows.

Problem (III): Find $(\mathbf{u}_h^n, p_h^n, T_h^n)\in X_h\times M_h\times W_h$ such that for $1\leqslant n\leqslant N$,

$$\begin{aligned}
&(\mathbf{u}_h^n,\mathbf{v}_h)+ka(\mathbf{u}_h^n,\mathbf{v}_h)+ka_1(\mathbf{u}_h^n,\mathbf{u}_h^n,\mathbf{v}_h)-kb(p_h^n,\mathbf{v}_h)+kb(q_h,\mathbf{u}_h^n)\\
&\quad+\sum_{K\in\Im_h}\delta_K(\mathbf{u}_h^n-k\nu\Delta\mathbf{u}_h^n+k(\mathbf{u}_h^n\cdot\nabla)\mathbf{u}_h+k\nabla p_h^n,\mathbf{v}_h-k\nu\Delta\mathbf{v}_h+k(\mathbf{u}_h^n\cdot\nabla)\mathbf{v}_h+k\nabla q_h)_K\\
&=\sum_{K\in\Im_h}\delta_K(kjT_h^n+\mathbf{u}_h^{n-1},\mathbf{v}_h-k\nu\Delta\mathbf{v}_h+k(\mathbf{u}_h^n\cdot\nabla)\mathbf{v}_h+k\nabla q_h)_K\\
&\quad+k(jT_h^n,\mathbf{v}_h)+(\mathbf{u}_h^{n-1},\mathbf{v}_h)\quad\forall(\mathbf{v}_h,q_h)\in X_h\times M_h
\end{aligned}\tag{4}$$

$$(T_h^n,\phi_h)+kD(T_h^n,\phi_h)+ka_2(\mathbf{u}_h^n,T_h^n,\phi_h)=(T_h^{n-1},\phi_h)\quad\forall\phi_h\in W_h$$

$$\mathbf{u}_h^0=\mathbf{0},\quad T_h^0=\psi(x,y),\quad (x,y)\in\Omega$$

where $\delta_K=\alpha h_K$, $\alpha>0$ is arbitrary constant.

Write $\hat{\mathbf{v}}=(\mathbf{v},p)$ and $\hat{\mathbf{w}}=(\mathbf{w},q)$. Define

$$\begin{aligned}
B_\delta(\mathbf{u},\mathbf{u}_h^n;\hat{\mathbf{v}},\hat{\mathbf{w}})&=(\mathbf{v},\mathbf{w})+ka(\mathbf{v},\mathbf{w})+ka_1(\mathbf{u},\mathbf{v},\mathbf{w})-kb(p,\mathbf{w})+kb(q,\mathbf{v})\\
&\quad+\sum_{K\in\Im_h}\delta_K(\mathbf{v}-k\nu\Delta\mathbf{v}+k(\mathbf{u}\cdot\nabla)\mathbf{v}+k\nabla p,\mathbf{w}-k\nu\Delta\mathbf{w}+k(\mathbf{u}_h^n\cdot\nabla)\mathbf{w}+k\nabla q)_K\\
F_{T\delta}(\hat{\mathbf{w}})&=k(jT_h^n,\mathbf{w})+(\mathbf{u}_h^{n-1},\mathbf{w})\\
&\quad+\sum_{K\in\Im_h}\delta_K(\mathbf{u}_h^{n-1}+kjT_h^n,\mathbf{w}-k\nu\Delta\mathbf{w}+k(\mathbf{u}_h^n\cdot\nabla)\mathbf{w}+k\nabla q)_K\\
\tilde{D}(\mathbf{v};T,\phi)&=(T,\phi)+kD(T,\phi)+ka_2(\mathbf{v},T,\phi)
\end{aligned}\tag{5}$$

Then Problem (III) could be rewritten as follows.

Problem (IV): Find $\hat{\mathbf{u}}_h^n\equiv(\mathbf{u}_h^n,p_h^n)\in X_h\times M_h$ such that, for $1\leqslant n\leqslant N$,

$$\begin{gathered}
B_\delta(\mathbf{u}_h^n,\mathbf{u}_h^n;\hat{\mathbf{u}}_h^n,\hat{\mathbf{w}}_h)=F_{T\delta}(\hat{\mathbf{w}}_h)\quad\forall\hat{\mathbf{w}}_h\equiv(\mathbf{v}_h,q_h)\in X_h\times M_h\\
\tilde{D}(\mathbf{u}_h^n;T_h^n,\phi_h)=(T_h^{n-1},\phi_h)\quad\forall\phi_h\in W_h\\
\mathbf{u}_h^0=\mathbf{0},\quad T_h^0=\psi(x,y)\ \text{ in }\Omega
\end{gathered}\tag{6}$$

where $\delta|_K=\delta_k$.

The following properties for trilinear forms $a_1(\cdot,\cdot,\cdot)$ and $a_2(\cdot,\cdot,\cdot)$ are often used (see [37]):

$$\begin{aligned}
&a_1(\mathbf{u},\mathbf{v},\mathbf{w})=-a_1(\mathbf{u},\mathbf{w},\mathbf{v}),\quad a_1(\mathbf{u},\mathbf{v},\mathbf{v})=0\quad\forall\mathbf{u},\mathbf{v},\mathbf{w}\in X\\
&a_2(\mathbf{u},T,\phi)=-a_2(\mathbf{u},\phi,T),\quad a_2(\mathbf{u},\phi,\phi)=0\quad\forall\mathbf{u}\in X,\ \forall T,\phi\in W
\end{aligned}\tag{7}$$

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where C_1 is a constant independent of u, v, and w, and C_2 is a constant independent of u, T, and ϕ. The bilinear forms $a(\cdot,\cdot)$, $D(\cdot,\cdot)$, and $b(\cdot,\cdot)$ have the following properties:

$$a(\mathbf{v},\mathbf{v}) \geqslant \nu|\mathbf{v}|_1^2 \quad \forall \mathbf{v} \in X, \quad |a(\mathbf{u},\mathbf{v})| \leqslant \nu|\mathbf{u}|_1|\mathbf{v}|_1 \quad \forall \mathbf{u},\mathbf{v} \in X \tag{8}$$

$$D(\phi,\phi) \geqslant \gamma_0^{-1}|\phi|_1^2 \quad \forall \phi \in W, \quad |D(T,\phi)| \leqslant \gamma_0^{-1}|T|_1|\phi|_1 \quad \forall T,\phi \in W \tag{9}$$

$$\sup_{\mathbf{v}\in X} \frac{b(q,\mathbf{v})}{|\mathbf{v}|_1} \geqslant \beta\|q\|_0 \quad \forall q \in M \tag{10}$$

where β is a constant. Define

$$N_0 = \sup_{\mathbf{u},\mathbf{v},\mathbf{w}\in X} \frac{a_1(\mathbf{u},\mathbf{v},\mathbf{w})}{|\mathbf{u}|_1\cdot|\mathbf{v}|_1\cdot|\mathbf{w}|_1}, \quad \tilde{N}_0 = \sup_{\mathbf{u}\in X,(T,\phi)\in W\times W} \frac{a_2(\mathbf{u},T,\phi)}{|\mathbf{u}|_1\cdot|T|_1\cdot|\phi|_1} \tag{11}$$

The following discrete Gronwall lemma is well known and very useful in the context of next analysis (see [4, 34]).

Lemma 2.2
If $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ are three non-negative sequences and $\{c_n\}$ is monotone, then they satisfy

$$a_n + b_n \leqslant c_n + \bar{\lambda}\sum_{i=0}^{n-1} a_i, \quad \bar{\lambda} > 0, \quad a_0 + b_0 \leqslant c_0$$

then

$$a_n + b_n \leqslant c_n \exp(n\bar{\lambda}), \quad n \geqslant 0$$

For Problem (III) or (IV), we have the following result [13].

Theorem 2.3
If h and k are sufficiently small and $h = O(k)$, then there exists $h_0 > 0$ such that when $h < h_0$ Problem (III) has a unique solution sequence $(\mathbf{u}_h^n, p_h^n, T_h^n) \in X_h \times M_h \times W_h$ and for $1 \leqslant n \leqslant N$,

$$\|\mathbf{u}_h^n\|_0^2 + k\sum_{i=1}^n \|\mathbf{u}_h^i\|_1^2 + \sum_{i=1}^n \|\delta^{1/2}(\mathbf{u}_h^i - k\nu\Delta\mathbf{u}_h^i + k(\mathbf{v}_h^i\cdot\nabla)\mathbf{u}_h^i + k\nabla p_h^i)\|_{0,h}^2 \leqslant RM \tag{12}$$

$$\|\mathbf{u}^n - \mathbf{u}_h^n\|_0 + (k\nu)^{1/2}\sum_{i=1}^n |\mathbf{u}^i - \mathbf{u}_h^i|_1 + k^{1/2}\sum_{i=1}^n \|p^i - p_h^i\|_0 + \|T^n - T_h^n\|_0$$

$$+ (k\gamma_0^{-1})^{1/2}\sum_{i=1}^n |T^i - T_h^i|_1 \leqslant C(h^\ell + h^\kappa + h^\iota) \tag{13}$$

where $M = t_N(R + 2kh\alpha)\|\varphi(x,y)\|_0^2 \exp(2\alpha h t_N)$, $R = \nu^{-1}$, $\|\cdot\|_{0,h}^2 = \sum_{K\in\Im_h}\|\cdot\|_{0,K}^2$, $(\hat{u}^n, T^n) = (\mathbf{u}^n, p^n, T^n) \in [W_0^{1,\infty}(\Omega)\cap H^{\ell+1}(\Omega)]^2 \times H^{\kappa+1}(\Omega) \times [W_0^{1,\infty}(\Omega)\cap H^{\ell+1}(\Omega)]$ are the solutions for Problem (II), and C is the constant dependent on $|\mathbf{u}^n|_{\ell+1}$, $|p^n|_\kappa$, and $|T^n|_{\iota+1}$.

Combining Theorems 2.1 and 2.3 yields the following result.

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Theorem 2.4
Under the assumptions of Theorems 2.1 and 2.3, there are the following error estimates, for $1\leqslant n\leqslant N$:

$$\|\mathbf{u}(t_n)-\mathbf{u}_h^n\|_0+(kv)^{1/2}\sum_{i=1}^{n}|\mathbf{u}(t_i)-\mathbf{u}_h^i|_1+k^{1/2}\sum_{i=1}^{n}\|p(t_i)-p_h^i\|_0$$
$$+\|T(t_n)-T_h^n\|_0+(k\gamma_0^{-1})^{1/2}\sum_{i=1}^{n}|T(t_i)-T_h^i|_1\leqslant C(k+h^\ell+h^\kappa+h^\imath)$$

If Re, Pr, triangulation parameter h, finite elements X_h, M_h, and W_h, and the time step increment k are all given, by solving Problem (III), we can obtain a solution ensemble $\{u_{1h}^n, u_{2h}^n, p_h^n, T_h^n\}_{n=1}^N$. And then we choose L (for example, $L=20$, $N=200$, in general, $L\ll N$) instantaneous solutions $\mathbf{U}_i(x, y)=(u_{1h}^{n_i}, u_{2h}^{n_i}, p_h^{n_i}, T_h^{n_i})(1\leqslant n_1<n_2<\cdots<n_L\leqslant N)$ (which are empirically elected and are useful and of interest for us to solve actual problem) from the N groups of solutions $(u_{1h}^n, u_{2h}^n, p_h^n, T_h^n)$ $(1\leqslant n\leqslant N)$ for Problem (III), which are referred to as snapshots.

3. OPTIMIZING REDUCED PLSMFE FORMULATION-BASED POD TECHNIQUE FOR THE NON-STATIONARY CONDUCTION–CONVECTION PROBLEMS

In this section, we use the POD technique to deal with the snapshots in Section 2 and to develop an optimizing reduced PLSMFE formulation for the non-stationary conduction–convection problems.

Let $\hat{X}=X\times M\times W$. For $\mathbf{U}_i(x, y)=(u_{1h}^{n_i}, u_{2h}^{n_i}, p_h^{n_i}, T_h^{n_i})$ $(i=1,2,\ldots,L)$ in Section 2, we set

$$\mathscr{V}=\mathrm{span}\{\mathbf{U}_1, \mathbf{U}_2, \ldots, \mathbf{U}_L\} \tag{14}$$

and refer to $\mathscr{V}$ as the ensemble consisting of the snapshots $\{\mathbf{U}_i\}_{i=1}^L$ at least one of which is supposed to be non-zero. Let $\{\boldsymbol{\psi}_j\}_{j=1}^l$ (where $\boldsymbol{\psi}_j=(\boldsymbol{\psi}_{uj}, \psi_{pj}, \psi_{Tj})$) denote an orthogonal basis of $\mathscr{V}$ with $l=\dim\mathscr{V}$. Then each member of the ensemble can be expressed as

$$\mathbf{U}_i=\sum_{j=1}^{l}(\mathbf{U}_i, \boldsymbol{\psi}_j)_{\hat{X}}\boldsymbol{\psi}_j \quad \text{for } i=1,2,\ldots,L \tag{15}$$

where $(\mathbf{U}_i, \boldsymbol{\psi}_j)_{\hat{X}}=(\nabla\mathbf{u}_h^{n_i}, \nabla\boldsymbol{\psi}_{uj})+(p_h^{n_i}, \psi_{pj})+(\nabla T_h^{n_i}, \nabla\psi_{Tj})$, $(\cdot,\cdot)$ is L^2-inner product, and $\boldsymbol{\psi}_{uj}$, ψ_{pj}, and ψ_{Tj} are orthogonal bases corresponding to u, p, and T, respectively.

The POD method consists in finding an orthogonal basis such that for every d $(1\leqslant d\leqslant l)$ the mean square error between the elements $\mathbf{U}_i$ $(1\leqslant i\leqslant L)$ and corresponding dth partial sum of (15) is minimized on average:

$$\min_{\{\psi_j\}_{j=1}^d}\frac{1}{L}\sum_{i=1}^{L}\left\|\mathbf{U}_i-\sum_{j=1}^{d}(\mathbf{U}_i, \boldsymbol{\psi}_j)_{\hat{X}}\boldsymbol{\psi}_j\right\|_{\hat{X}}^2 \tag{16}$$

such that

$$(\boldsymbol{\psi}_i, \boldsymbol{\psi}_j)_{\hat{X}}=\delta_{ij} \quad \text{for } 1\leqslant i\leqslant d,\ \ 1\leqslant j\leqslant i \tag{17}$$

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where $\|\mathbf{U}_i\|_{\hat{X}}=[\|\nabla u_{1h}^{n_i}\|_0^2+\|\nabla u_{2h}^{n_i}\|_0^2+\|p_h^{n_i}\|_0^2+\|\nabla T_h^{n_i}\|_0^2]^{1/2}$. A solution $\{\boldsymbol{\psi}_j\}_{j=1}^d$ of (16) and (17) is known as a POD basis of rank d. Note that $\|u\|_1$ is equivalent to $\|\nabla u\|_0$ for $u\in H_0^1(\Omega)$, which show that inner product and norm including only the gradient of the function are reasonable.

We introduce the correlation matrix $\mathbf{G}=(G_{ij})_{L\times L}\in R^{L\times L}$ corresponding to the snapshots $\{\mathbf{U}_i\}_{i=1}^L$ by

$$G_{ij}=\frac{1}{L}(\mathbf{U}_i,\mathbf{U}_j)_{\hat{X}} \tag{18}$$

The matrix $\mathbf{G}$ is positive semi-definite and has rank l. The solution of (16) and (17) can be found in [16, 19, 31], for example.

Proposition 3.1
Let $\lambda_1\geqslant\lambda_2\geqslant\cdots\geqslant\lambda_l>0$ denote the positive eigenvalues of $\mathbf{G}$ and $\mathbf{v}_1,\mathbf{v}_2,\ldots,\mathbf{v}_l$ the associated eigenvectors. Then a POD basis of rank $d\leqslant l$ is given as

$$\boldsymbol{\psi}_i=\frac{1}{\sqrt{\lambda_i}}\mathbf{v}_i^{\mathrm{T}}(\mathbf{U}_1,\mathbf{U}_2,\ldots,\mathbf{U}_L)^{\mathrm{T}}=\frac{1}{\sqrt{\lambda_i}}\sum_{j=1}^{L}(\mathbf{v}_i)_j\mathbf{U}_j \tag{19}$$

where $(\mathbf{v}_i)_j$ denotes the jth component of the eigenvector $\mathbf{v}_i$. Furthermore, the following error formula holds:

$$\frac{1}{L}\sum_{i=1}^{L}\left\|\mathbf{U}_i-\sum_{j=1}^{d}(\mathbf{U}_i,\boldsymbol{\psi}_j)_{\hat{X}}\boldsymbol{\psi}_j\right\|_{\hat{X}}^2=\sum_{j=d+1}^{l}\lambda_j \tag{20}$$

Let $\mathscr{V}^d=\mathrm{span}\{\boldsymbol{\psi}_1,\boldsymbol{\psi}_2,\ldots,\boldsymbol{\psi}_d\}$ and $X^d\times M^d\times W^d=\mathscr{V}^d$ with $X^d\subset X$, $M^d\subset M$, and $W^d\subset W$. Let the Ritz-projection $P^d:X\rightarrow X^d$, L^2-projection $\rho^d:M\rightarrow M^d$, and the Ritz-projection $\varrho^d:W\rightarrow W^d$ be denoted by, respectively,

$$\begin{aligned} a(P^d\mathbf{u},\mathbf{v}_d)&=a(\mathbf{u},\mathbf{v}_d)\quad\forall\mathbf{v}_d\in X^d\\ (\rho^d p,q_d)&=(p,q_d)\quad\forall q_d\in M^d\\ D(\varrho^d w,w^d)&=D(w,w^d)\quad\forall w_d\in W^d \end{aligned} \tag{21}$$

where $\mathbf{u}\in X$, $p\in M$, and $w\in W$. Owing to (21) the linear operators P^d, ρ^d, and ϱ^d are well defined and bounded:

$$\begin{aligned} \|\nabla(P^d\mathbf{u})\|_0&\leqslant\|\nabla\mathbf{u}\|_0\quad\forall\mathbf{u}\in X\\ \|\rho^d p\|_0&\leqslant\|p\|_0\quad\forall p\in M\\ \|\nabla(\varrho^d w)\|_0&\leqslant\|\nabla w\|_0\quad\forall w\in W \end{aligned} \tag{22}$$

Lemma 3.2
For every d $(1 \leqslant d \leqslant l)$ the projection operators P^d, ρ^d, and ϱ^d satisfy, respectively,

$$\begin{aligned} \frac{1}{L}\sum_{i=1}^{L}\|\nabla(\mathbf{u}_h^{n_i}-P^d\mathbf{u}_h^{n_i})\|_0^2 &\leqslant \sum_{j=d+1}^{l}\lambda_j \\ \frac{1}{L}\sum_{i=1}^{L}\|p_h^{n_i}-\rho^d p_h^{n_i}\|_0^2 &\leqslant \sum_{j=d+1}^{l}\lambda_j \\ \frac{1}{L}\sum_{i=1}^{L}\|\nabla(T_h^{n_i}-P^d T_h^{n_i})\|_0^2 &\leqslant \sum_{j=d+1}^{l}\lambda_j \end{aligned} \tag{23}$$

Proof
For any $u_h^{n_i} \in X_h$ $(i=1,2,\ldots,L)$ we deduce from (21) that

$$\begin{aligned} v\|\nabla(\mathbf{u}_h^{n_i}-P^d\mathbf{u}_h^{n_i})\|_0^2 &= a(\mathbf{u}_h^{n_i}-P^d\mathbf{u}_h^{n_i},\mathbf{u}_h^{n_i}-P^d\mathbf{u}_h^{n_i}) = a(\mathbf{u}_h^{n_i}-P^d\mathbf{u}_h^{n_i},\mathbf{u}_h^{n_i}-\mathbf{v}_d) \\ &\leqslant v\|\nabla(\mathbf{u}_h^{n_i}-P^d\mathbf{u}_h^{n_i})\|_0\|\nabla(\mathbf{u}_h^{n_i}-\mathbf{v}_d)\|_0 \quad \forall \mathbf{v}_d \in X^d \end{aligned}$$

Furthermore,

$$\|\nabla(\mathbf{u}_h^{n_i}-P^d\mathbf{u}_h^{n_i})\|_0 \leqslant \|\nabla(\mathbf{u}_h^{n_i}-\mathbf{v}_d)\|_0 \quad \forall \mathbf{v}_d \in X^d \tag{24}$$

Taking $\mathbf{v}_d=\sum_{j=1}^{d}(\nabla\mathbf{u}_h^{n_i},\nabla\boldsymbol{\psi}_{uj})\boldsymbol{\psi}_{uj}$ (where $\boldsymbol{\psi}_{uj}$ is the component of $\boldsymbol{\psi}_j$ corresponding to $\mathbf{u}$) in (24), we can obtain the first inequality of (23) from (20).

Using the Hölder inequality and the second equality of (21) yields

$$\begin{aligned} \|p_h^{n_i}-\rho^d p_h^{n_i}\|_0^2 &= (p_h^{n_i}-\rho^d p_h^{n_i}, p_h^{n_i}-\rho^d p_h^{n_i}) = (p_h^{n_i}-\rho^d p_h^{n_i}, p_h^{n_i}-q_d) \\ &\leqslant \|p_h^{n_i}-\rho^d p_h^{n_i}\|_0\|p_h^{n_i}-q_d\|_0 \quad \forall q_d \in M^d \end{aligned}$$

Consequently,

$$\|p_h^{n_i}-\rho^d p_h^{n_i}\|_0 \leqslant \|p_h^{n_i}-q_d\|_0 \quad \forall q_d \in M^d \tag{25}$$

Taking $q_d=\sum_{j=1}^{d}(p_h^{n_i},\psi_{pj})_0\psi_{pj}$ (where ψ_{pj} is the component of $\boldsymbol{\psi}_j$ corresponding to p) in (25), from (20) we can obtain the second inequality of (23).

Using the same technique as the first inequality of (23) can prove the third inequality of (23), which completes the proof of Lemma 3.2. □

Thus, using $\mathscr{V}^d = X^d \times M^d \times W^d$, we can obtain the optimizing reduced PLSMFE formulation for Problem (IV) as follows.

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Problem (V): Find $(\hat{\mathbf{u}}_d^n, T_d^n) \equiv (\mathbf{u}_d^n, p_d^n, T_d^n) \in \mathscr{V}^d$ such that

$$\begin{aligned} B_\delta(\mathbf{u}_d^n, \mathbf{u}_d^n; \hat{\mathbf{u}}_d^n, \hat{\mathbf{w}}_d) &= F_{T\delta}(\hat{\mathbf{w}}_d) \quad \forall \hat{\mathbf{w}}_d^n \equiv (\mathbf{v}_d, q_d) \in X^d \times M^d \\ \tilde{D}(\mathbf{u}_d^n; T_d^n, \phi_d) &= (T_d^{n-1}, \phi_d) \quad \forall \phi_d \in W^d \\ \mathbf{u}_d^0 = \mathbf{0}, &\quad T_d^0 = \psi(x,y), \quad (x,y) \in \Omega \end{aligned} \tag{26}$$

where $1 \leqslant n \leqslant N$.

Remark 2
Problem (V) is an optimizing reduced PLSMFE formulation based on POD technique for Problem (IV), since it only includes $4d$ degrees of freedom, whereas Problem (IV) includes $4N_p$ if $\kappa = \ell = \imath = 1$ (where N_p is the number of the vertices in $\Im_h$) and also includes $4N_p + 4N_s \approx 16N_p$ if $\kappa = \ell = \imath = 2$ and $4d \ll 4N_p \ll 16N_p$ (where N_s is the number of the sides in $\Im_h$). And since the residual $\delta_K^{1/2}(\mathbf{u}_d^i - k\nu\Delta\mathbf{u}_d^n + k(\mathbf{v}_d^n\cdot\nabla)\mathbf{u}_d^n + k\nabla p_d^n - kjT_d^n - \mathbf{u}_d^{n-1})_K$ in Problem (V) is introduced, the combination of finite element subsets need not satisfy the BB stability condition and optimizing-order error estimates can be obtained (see Section 4). When one computes real-life problems, one may obtain the ensemble of snapshots from physical system trajectories by drawing samples from experiments and interpolation (or data assimilation). For example, for weather forecast, one can use previous weather prediction results to construct the ensemble of snapshots, then restructure the POD basis for the ensemble of snapshots by above (16)–(19), and finally combine it with a Petrov–Galerkin least squares projection to derive an optimizing reduced-order dynamical system, i.e. one needs only to solve Problem (V) with few degrees of freedom, without having to solve Problem (IV). Thus, a forecast of future weather change can be simulated in a fast manner, which is of major importance for actual real-life applications.

4. EXISTENCE AND ERROR ANALYSIS OF SOLUTIONS OF THE OPTIMIZING REDUCED PLSMFE FORMULATION

This section is devoted to discussing the existence and error estimates of solutions for Problem (V).

We see from (19) that $\mathscr{V}^d = X^d \times M^d \times W^d \subset \mathscr{V} \subset X_h \times M_h \times W_h \subset X \times M \times W$.

We first obtain the following existence result for solutions of Problem (V), whose proof is provided in Appendix A.

Theorem 4.1
Under the assumptions of Theorems 2.1 and 2.3, Problem (V) has a unique solution sequence $(\mathbf{u}_d^n, p_d^n, T_d^n) \in X^d \times M^d \times W^d$ and satisfies, for $1 \leqslant n \leqslant N$,

$$\left[\|\mathbf{u}_d^n\|_0^2 + k\sum_{i=1}^n \|\mathbf{u}_d^i\|_1^2 + \|\delta^{1/2}(\mathbf{u}_d^n - k\nu\Delta\mathbf{u}_d^n + k\mathbf{u}_d^n\nabla\mathbf{u}_d^n + k\nabla p_d^n)\|_{0,h}^2 \right]^{1/2} \leqslant \sqrt{RM} \tag{27}$$

In the following theorem, the error estimates between the solutions $(\mathbf{u}_d^n, p_d^n, T_d^n)$ for Problem (V) and the solutions $(\mathbf{u}_h^n, p_h^n, T_h^n)$ for Problem (IV) are derived, whose proof is provided in Appendix B.

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Theorem 4.2
Under the assumptions of Theorems 2.1 and 2.3, let $n_0=0$ and $N<n_{L+1}$, if h and k are sufficiently small, $h=O(k)$, and $k=O(L^{-2})$, then the errors between the solutions $(\mathbf{u}_d^n, p_d^n, T_d^n)$ for Problem (V) and the solutions $(\mathbf{u}_h^n, p_h^n, T_h^n)$ for Problem (IV) have the following error estimates, for $1\leqslant n\leqslant N$, if $n=n_i\in\{n_1,n_2,\ldots,n_L\}$,

$$\|\mathbf{u}_h^{n_i}-\mathbf{u}_d^{n_i}\|_0+\|T_h^{n_i}-T_d^{n_i}\|_0+(kv)^{1/2}\sum_{j=n_1}^{n_i}\|\nabla(\mathbf{u}_h^j-\mathbf{u}_d^j)\|_0+k^{1/2}\sum_{j=n_1}^{n_i}\|p_h^j-p_d^j\|_0$$
$$+(k\gamma_0^{-1})^{1/2}\sum_{j=n_1}^{n_i}\|\nabla(T_h^j-T_d^j)\|_0\leqslant C\left(k^{1/2}\sum_{j=d+1}^{l}\lambda_j\right)^{1/2} \tag{28}$$

and if snapshots are taken at uniform intervals, $n_i<n<n_{i+1}$ $(i=0,1,2,\ldots,L)$,

$$\|\mathbf{u}_h^n-\mathbf{u}_d^n\|_0+\|T_h^n-T_d^n\|_0+(kv)^{1/2}\left[\|\nabla(\mathbf{u}_h^n-\mathbf{u}_d^n)\|_0+\sum_{j=1}^{n_i}\|\nabla(\mathbf{u}_h^j-\mathbf{u}_d^j)\|_0\right]$$
$$+(k\gamma_0^{-1})^{1/2}\left[\|\nabla(T_h^n-T_d^n)\|_0+\sum_{j=1}^{n_i}\|\nabla(T_h^j-T_d^j)\|_0\right]$$
$$+k^{1/2}\left[\|p_h^n-p_d^n\|_0+\sum_{j=1}^{n_i}\|p_h^j-p_d^j\|_0\right]\leqslant Ck+C\left(k^{1/2}\sum_{j=d+1}^{l}\lambda_j\right)^{1/2} \tag{29}$$

Combining Theorems 2.4 and 4.2 yields the following result.

Theorem 4.3
Under Theorems 2.4 and 4.2 hypotheses, the error estimates between the solutions $(u(t), p(t), T(t))$ for Problem (I) and the solutions $(\mathbf{u}_d^n, p_d^n, T_d^n)$ for the reduced-order basic Problem (V) are, for $n=1,2,\ldots,N$, if $n=n_i\in\{n_1,n_2,\ldots,n_L\}$,

$$\|\mathbf{u}(t_{n_i})-\mathbf{u}_d^{n_i}\|_0+\|T(t_{n_i})-T_d^{n_i}\|_0+(kv)^{1/2}\sum_{j=n_1}^{n_i}\|\nabla(\mathbf{u}^j-\mathbf{u}_d^j)\|_0+k^{1/2}\sum_{j=n_1}^{n_i}\|p^j-p_d^j\|_0$$
$$+(k\gamma_0^{-1})^{1/2}\sum_{j=n_1}^{n_i}\|\nabla(T^j-T_d^j)\|_0\leqslant C(h^\kappa+h^\ell+h^\iota+k)+C\left(k^{1/2}\sum_{j=d+1}^{l}\lambda_j\right)^{1/2}$$

and if snapshots are taken at uniform intervals, $n_i<n<n_{i+1}$ $(i=0,1,2,\ldots,L)$,

$$\|\mathbf{u}(t_n)-\mathbf{u}_d^n\|_0+\|T(t_n)-T_d^n\|_0+(kv)^{1/2}\left[\|\nabla(\mathbf{u}(t_n)-\mathbf{u}_d^n)\|_0+\sum_{j=n_1}^{n_i}\|\nabla(\mathbf{u}(t_j)-\mathbf{u}_d^j)\|_0\right]$$
$$+(k\gamma_0^{-1})^{1/2}\left[\|\nabla(T(t_n)-T_d^n)\|_0+\sum_{j=n_1}^{n_i}\|\nabla(T(t_j)-T_d^j)\|_0\right]$$
$$+k^{1/2}\left[\|p(t_n)-p_d^n\|_0+\sum_{j=n_1}^{n_i}\|p(t_j)-p_d^j\|_0\right]\leqslant C(h^\kappa+h^\ell+h^\iota+k)+C\left(k^{1/2}\sum_{j=d+1}^{l}\lambda_j\right)^{1/2}$$

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Remark 3

The condition $k=O(L^{-2})$ in Theorems 4.2 and 4.3 implies $N=O(L^2)$, which shows the relation between the number L of snapshots and the number N of all time instances. Therefore, it is unnecessary to take the total number of transient solutions at all time instances t_n as snapshots, for instance in [30, 31]; instead it is sufficient to take one snapshot every 10 time intervals. Theorems 4.2 and 4.3 have presented the error estimates between the solutions of the optimizing reduced PLSMFE formulation Problem (V) and the solutions of usual PLSMFE formulation Problems (III) and (I), respectively. Since our methods employ some usual PLSMFE solutions $(\mathbf{u}_h^{n_i}, p_h^{n_i}, T_h^{n_i})$ $(i=1,2,\ldots,L)$ for Problem (III) as assistant analysis (see Appendix B), the error estimates in Theorem 4.3 are correlated to the spatial grid scale h and the time step size k. However, when one computes real-life problems, one may obtain the ensemble of snapshots from the physical system trajectories by drawing samples from experiments and interpolation (or data assimilation). Thus, the PLSMFE solutions $(\mathbf{u}_h^{n_i}, p_h^{n_i}, T_h^{n_i})$ $(i=1,2,\ldots,L)$ could be replaced by the interpolation functions of experimental and previous results, avoiding solving full-order basic Problem (III) and requiring only to solve directly Problem (V), which includes very few degrees of freedom since it depends only on d $(d \ll l \leqslant L \ll N)$, in general. Since the development and change of numerous future nature phenomena are closely related to previous results (for example, weather change, biology anagenesis, and so on), using existing results as snapshots to structure POD basic, by solving corresponding PDEs one will well and truly capture future law of the development and change of natural phenomena. Therefore, these POD methods are of valuable for important applications. If $\mathscr{V}^d \subset H^2(\Omega)^2$, Problem (V) will not appear $\sum_{K\in\Im_h}$, i.e. Problem (V) will be independent of K, i.e. it is independent of the spatial grid scale h.

5. SOME NUMERICAL EXPERIMENTS

In this section, we present some numerical examples with a physical model of cavity flow for second-order element (i.e. $\ell=\kappa=\iota=2$) and with different Reynolds numbers by the optimizing the reduced PLSMFE formulation Problem (V) validating our theoretical results.

Let the side length of the cavity be 1 (see Figure 1). We first divide the cavity into $32\times 32=1024$ small squares with side length $\Delta x=\Delta y=\frac{1}{32}$, and then link diagonal of square to divide each square into two triangles in the same direction, which consists of triangularization $\Im_h$ $(h=\sqrt{2}/32)$. We take a time step increment as $\Delta t=0.001$. Let the initial value and boundary values of u and v be equal to 0 on boundary of the cavity are also taken as 0. And let $T=0$ on left and lower boundary of the cavity, $\partial T/\partial y=0$ on upper boundary of the cavity, and $T=4y(1-y)$ on right boundary of the cavity (see Figure 1). Put $Pr=0.71$ and $Re=2000$ or 5000.

We obtain 20 values (i.e. snapshots) outputting at time $t=10, 20, 30, \ldots, 200$ by solving classical PLSMFE formulation, i.e. Problem (IV). It is numerically shown that the eigenvalues satisfy $[k^{1/2}\sum_{i=6}^{20}\lambda_i]^{1/2}\leqslant 2\times 10^{-3}$. When $t=200$, we obtain the solutions of the reduced formulation Problem (IV) based on the POD method of MEF depicted graphically in Figures 2–7 on right-hand side using five optimal POD bases if $Re=2000$ and 5000, whereas the solutions obtained with classical PLSMFE formulation Problem (III) are depicted graphically in Figures 2–7 on left-hand side. (Since these figures are equal to solutions obtained with 20 bases, they are also known as the figures of solution using full bases.)

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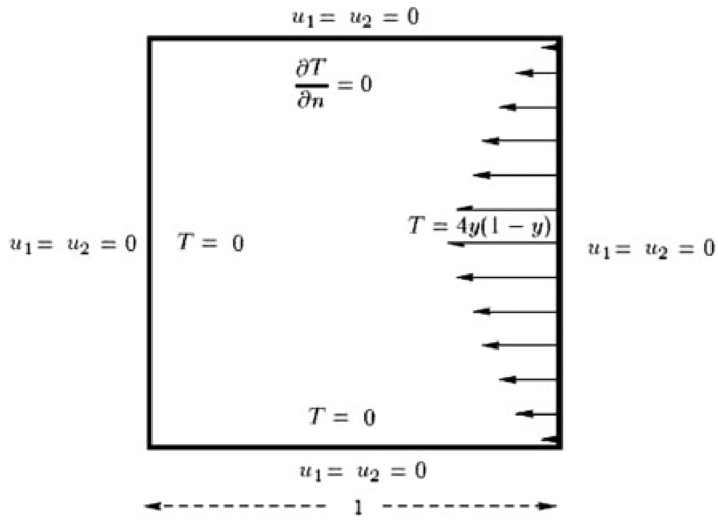

Figure 1. Physics model of the cavity flows: $t=0$, i.e. $n=0$ initial values on boundary.

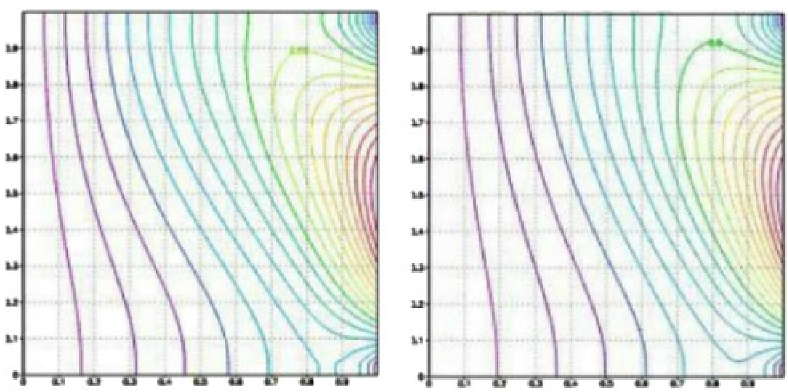

Figure 2. For $Re=2000$, temperature figure for classical PLSMFE solution (figure on left-hand side) and when $d=5$ the optimizing reduced PLSMFE solution (figure on right-hand side).

Figure 8 shows the errors between solutions obtained with different numbers of optimal POD bases and solutions obtained with the full bases. Comparing classical PLSMFE formulation Problem (III) with the reduced PLSMFE formulation Problem (V) containing only five optimal bases implementing 3000 times numerical simulation computations, we find that for the classical implementation for PLSMFE formulation Problem (III) the performing time required is 12 min, whereas for the optimizing reduced PLSMFE formulation Problem (V) with five optimal bases the required performing time is only 3 s, i.e. the classical PLSMFE formulation Problem (III) required performing time that is by a factor of 240 larger than the optimizing reduced PLSMFE formulation Problem (V) with five optimal bases required performing time, whereas the errors between their respective solutions do not exceed 3×10^{-3}. Although our examples are in sense recomputing what

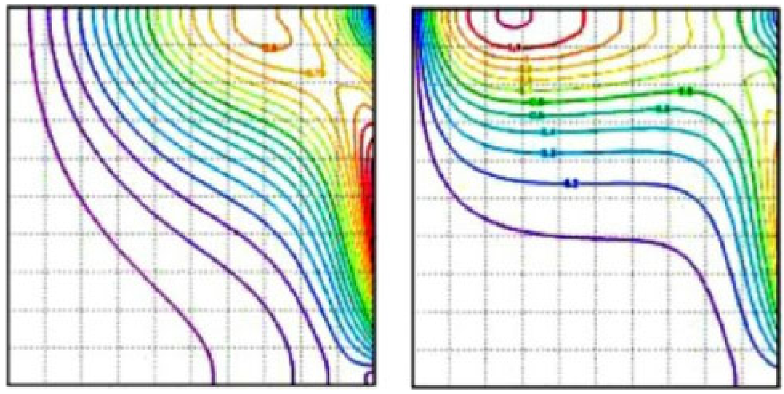

Figure 3. For $Re = 5000$, temperature figure for classical PLSMFE solution (figure on left-hand side) and when $d = 5$ the optimizing reduced PLSMFE solution (figure on right-hand side).

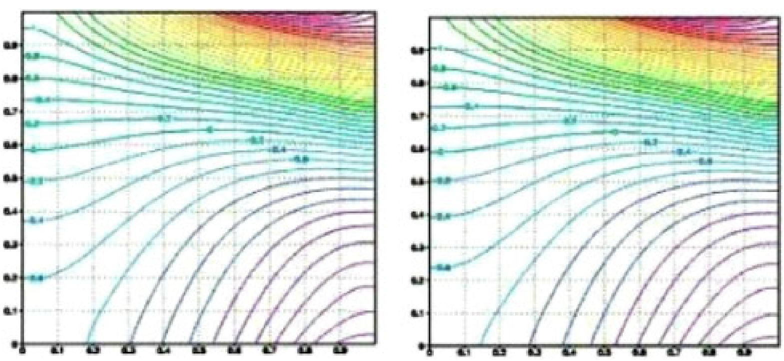

Figure 4. For $Re = 2000$, pressure figure for classical PLSMFE solution (figure on left-hand side) and when $d = 5$ the optimizing reduced PLSMFE solution (figure on right-hand side).

we have already computed by usual PLSMFE formulation Problem (III), when we compute actual problems, we may structure the snapshots and POD basis with interpolation or data assimilation by drawing samples from experiments, then solve directly Problem (V), while it is unnecessary to solve Problem (III); thus, the time-consuming calculations and resource demands in the computational process will be greatly reduced. It is also shown that finding the approximate solutions for the non-stationary conduction–convection problems with the optimizing reduced PLSMFE formulation Problem (V) is very effective. In addition, the results obtained for the numerical examples are consistent with the theoretical ones.

6. CONCLUSIONS

In this paper, we have employed the POD techniques to derive an optimizing reduced PLSMFE formulation for the non-stationary conduction–convection problems (see Remark 2). We first

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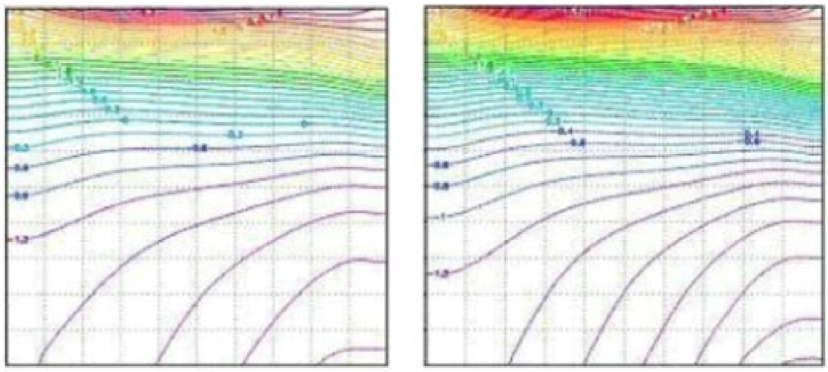

Figure 5. For $Re=5000$, pressure stream line figure for classical PLSMFE solutions (figure on left-hand side) and when $d=5$, the optimizing reduced PLSMFE solution (figure on right-hand side).

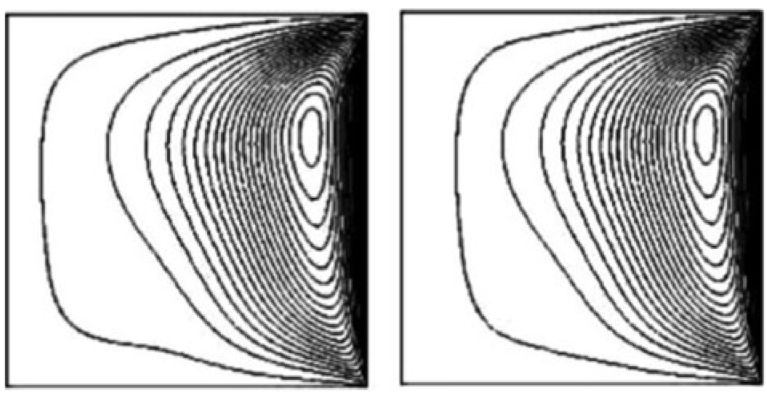

Figure 6. For $Re=2000$, velocity stream line figure for classical PLSMFE solutions (figure on left-hand side) and when $d=5$ the optimizing reduced PLSMFE solution (figure on right-hand side).

reconstruct optimal orthogonal bases of ensembles of data compiled from transient solutions derived by using the usual PLSMFE equation system, whereas in actual applications, one may obtain the ensemble of snapshots from physical system trajectories by drawing samples from experiments and interpolation (or data assimilation). We have also combined the optimal orthogonal bases with a Petrov–Galerkin least squares projection procedure, thus yielding a new optimizing reduced PLSMFE formulation of lower-dimensional order and of sufficient accuracy for the non-stationary conduction–convection problems. We have then proceeded to derive error estimates between our optimizing reduced PLSMFE approximate solutions and the usual PLSMFE approximate solutions and have shown using numerical examples that the errors between the optimizing reduced PLSMFE approximate solutions and the usual PLSMFE solutions are consistent with the theoretical error results. Since this paper is already too long, the analysis of conditioning of our POD-reduced PLSMFE formulation compared with the usual PLSMFE formulation and the problems in 3D

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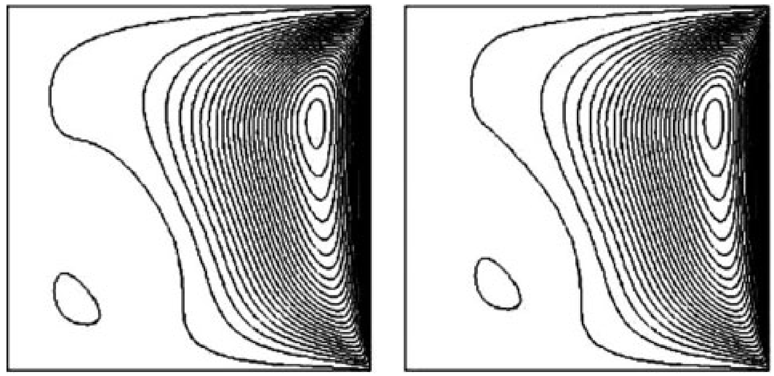

Figure 7. For $Re=5000$, velocity stream line figure for classical PLSMFE solutions (figure on left-hand side) and when $d=5$ the optimizing of the reduced PLSMFE solution (figure on right-hand side).

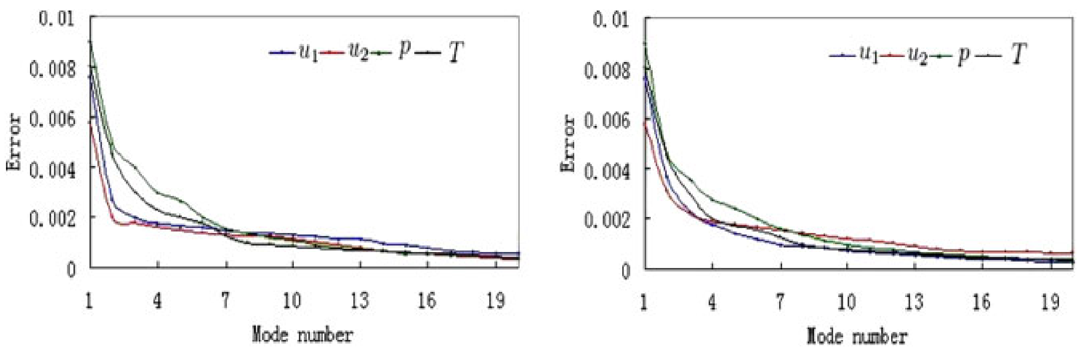

Figure 8. Error for $Re=2000$ on left–hand side and error for $Re=5000$ on right-hand side.

simulations are not included and will be further investigated in a new paper in advanced stages of completion. These issues are of high current research interest. Future research work in this area aims at addressing more complicated PDEs, extending the optimizing reduced PLSMFE formulation, applying it to a realistic atmospheric operational forecast system and to a set of more complicated nonlinear PDEs, for instance, 3D realistic model equations coupling strongly nonlinear properties, non-homogeneous variable flux and boundary, etc. Using theoretical analysis and numerical examples, we have shown that the optimizing reduced PLSMFE formulation presented herein has extensive potential applications.

APPENDIX A

The proof of Theorem 4.1 is as follows.

We use Brouwer's fixed point theorem to prove Theorem 4.1.

For all $\mathbf{v}_d^n \in X_d$ and $\|\mathbf{v}_d^n\|_0^2 + k\sum_{i=1}^n \|\mathbf{v}_d^i\|_1^2 \leqslant RM$, consider the following linearized problem:

$$\begin{aligned} B_\delta(\mathbf{v}_d^n, \mathbf{v}_d^n; \hat{\mathbf{u}}_d^n, \hat{\mathbf{w}}_d) &= F_\delta(\hat{\mathbf{w}}_d) \quad \forall \hat{\mathbf{w}}_d \in X^d \times M^d \\ \mathbf{u}_d^0 &= \mathbf{0} \quad \text{in } \Omega \end{aligned} \tag{A1}$$

$$\begin{aligned} \tilde{D}(\mathbf{v}_d^n; T_d^n, \phi_d) &= (T_d^{n-1}, \phi_d) \quad \forall \phi_d \in W^d \\ T_d^0 &= \psi(x, y) \quad \text{in } \Omega \end{aligned} \tag{A2}$$

Since $\tilde{D}(\mathbf{v}_d^n; \cdot, \cdot)$ is a coercive bilinear functional, linearized problem (A2) has a unique group of solutions $T_d^n \in W^d$ $(n=1,2,\ldots,N)$. For known T_d^n, since $B_\delta(\mathbf{v}_d^n, \mathbf{v}_d^n; \cdot, \cdot)$ is a coercive bilinear functional, linearized problem (A1) has a unique group of solutions $\hat{u}_d^n = (\mathbf{u}_d^n, p_d^n)$ $(n=1,2,\ldots,N)$. Thus, there exists a map $G: (\hat{\mathbf{v}}_d^n, \chi_d^n) \to (\hat{\mathbf{u}}_d^n, T_d^n)$ $(n=1,2,\ldots,N)$, where $\hat{\mathbf{v}}_d^n = (\mathbf{v}_d^n, \psi_d^n)$.

Taking $\phi_d = T_d^n$ in (A2), from (7) we obtain

$$\|T_d^n\|_0^2 + 2\gamma_0^{-1} k \|\nabla T_d^n\|_0^2 \leqslant \|T_d^{n-1}\|_0^2 \tag{A3}$$

Summing (A3) from 1 to n yields

$$\|T_d^n\|_0^2 + \gamma_0^{-1} k \sum_{i=1}^n \|\nabla T_d^i\|_0^2 \leqslant \|\psi(x,y)\|_0^2 \tag{A4}$$

Taking $\hat{\mathbf{w}}_d = \hat{\mathbf{u}}_d^n$ in (A1), we obtain

$$\begin{aligned} &\|\mathbf{u}_d^n\|_0^2 + kv|\mathbf{u}_d^n|_1^2 + \|\delta^{1/2}(\mathbf{u}_d^n - kv\Delta\mathbf{u}_d^n + k(\mathbf{v}_d^n\cdot\nabla)\mathbf{u}_d^n + k\nabla p_d^n)\|_{0,h}^2 \\ &\quad = \sum_{K\in\Im_h} \delta_K (kjT_d^n + \mathbf{u}_d^{n-1}, \mathbf{u}_d^n - kv\Delta\mathbf{u}_d^n + k(\mathbf{v}_d^n\cdot\nabla)\mathbf{u}_d^n + k\nabla p_d^n)_K \\ &\qquad + kj(T_d^n, \mathbf{u}_d^n) + (\mathbf{u}_d^{n-1}, \mathbf{u}_d^n) \\ &\quad \leqslant \frac{1}{2}(kR\|T_d^n\|_{-1}^2 + kv|\mathbf{u}_d^n|_1^2) + \frac{1}{2}(\|\mathbf{u}_d^n\|_0^2 + \|\mathbf{u}_d^{n-1}\|_0^2) \\ &\qquad + \frac{1}{2}\|\delta^{1/2}(kT_d^n + \mathbf{u}_d^{n-1})\|_0^2 + \frac{1}{2}\|\delta^{1/2}(\mathbf{u}_d^n - kv\Delta\mathbf{u}_d^n + k(\mathbf{v}_d^n\cdot\nabla)\mathbf{u}_d^n + k\nabla p_d^n)\|_{0,h}^2 \end{aligned} \tag{A5}$$

Noting that $\|\cdot\|_{-1} \leqslant \|\cdot\|_0$. From (A5), we have

$$\begin{aligned} &\|\mathbf{u}_d^n\|_0^2 + kv|\mathbf{u}_d^n|_1^2 + \|\delta^{1/2}(\mathbf{u}_d^n - kv\Delta\mathbf{u}_d^n + k(\mathbf{v}_d^n\cdot\nabla)\mathbf{u}_d^n + k\nabla p_d^n)\|_{0,h}^2 \\ &\quad \leqslant kR\|T_d^n\|_0^2 + \|\mathbf{u}_d^{n-1}\|_0^2 + 2\|\delta^{1/2} kT_d^n\|_0^2 + 2\alpha h\|\mathbf{u}_d^{n-1}\|_0^2 \\ &\quad \leqslant k(R + 2kh\alpha)\|\varphi(x,y)\|_0^2 + \|\mathbf{u}_d^{n-1}\|_0^2 + 2\alpha h\|\mathbf{u}_d^{n-1}\|_0^2 \end{aligned} \tag{A6}$$

Summing (A6) from 1 to n and noting that $\mathbf{u}_d^0 = 0$ could yield

$$\begin{aligned} &\|\mathbf{u}_d^n\|_0^2 + kv\sum_{i=1}^n |\mathbf{u}_d^i|_1^2 + \sum_{i=1}^n \|\delta^{1/2}(\mathbf{u}_d^i - kv\Delta\mathbf{u}_d^i + k(\mathbf{v}_d^i\cdot\nabla)\mathbf{u}_d^i + k\nabla p_d^i)\|_{0,h}^2 \\ &\quad \leqslant nk(R + 2kh\alpha)\|\varphi(x,y)\|_0^2 + 2\alpha h\sum_{i=0}^{n-1} \|\mathbf{u}_d^i\|_0^2 \end{aligned} \tag{A7}$$

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By discrete Gronwall inequality, we obtain

$$\|\mathbf{u}_d^n\|_0^2+k\nu\sum_{i=1}^{n}|\mathbf{u}_d^i|_1^2+\sum_{i=1}^{n}\|\delta^{1/2}(\mathbf{u}_d^i-k\nu\Delta\mathbf{u}_d^i+k(\mathbf{v}_d^i\cdot\nabla)\mathbf{u}_d^i+k\nabla p_d^i)\|_{0,h}^2$$
$$\leqslant nk(R+2kh\alpha)\|\varphi(x,y)\|_0^2\exp(2\alpha hn) \tag{A8}$$

Note that $1<\nu^{-1}$ and $kn\leqslant t_N$. From (A8) we obtain

$$\|\mathbf{u}_d^n\|_0^2+k\sum_{i=1}^{n}|\mathbf{u}_d^i|_1^2+\sum_{i=1}^{n}\|\delta^{1/2}(\mathbf{u}_d^i-k\nu\Delta\mathbf{u}_d^i+k(\mathbf{v}_d^i\cdot\nabla)\mathbf{u}_d^i+k\nabla p_d^i)\|_{0,h}^2$$
$$\leqslant Rt_N(R+2kh\alpha)\|\varphi(x,y)\|_0^2\exp(2\alpha t_N)\equiv RM \tag{A9}$$

Let $B_{RM}=\{(\mathbf{v}_d^n,\psi_d^n,\chi_d^n)\in X^d\times M^d\times W^d;\ \|\mathbf{v}_d^n\|_0^2+k\sum_{i=1}^{n}|\mathbf{v}_d^i|_1^2\leqslant RM\}$. It is shown by (A4) and (A9) that the map $G:B_{RM}\to B_{RM}$. Thus, it is necessary to prove that F is continuous. For any $(\hat{\mathbf{v}}_d^{1n},\chi_d^{1n})=(\mathbf{v}_d^{1n},\psi_d^{1n},\chi_d^{1n})$ and $(\hat{\mathbf{v}}_d^{2n},\chi_d^{2n})=(\mathbf{v}_d^{2n},\psi_d^{2n},\chi_d^{2n})\in B_{RM}$, by (A1) and (A2), we obtain two groups of solutions $(\mathbf{u}_d^{1n},p_d^{1n},T_d^{1n})$ and $(\mathbf{u}_d^{2n},p_d^{2n},T_d^{2n})$ $(n=1,2,\dots,N)$ such that

$$\begin{aligned}
&B_\delta(\mathbf{v}_d^{1n},\mathbf{v}_d^{1n};\hat{\mathbf{u}}_d^{1n},\hat{\mathbf{w}}_d)=F_\delta(\hat{\mathbf{w}}_d)\quad\forall\hat{\mathbf{w}}_d\in X^d\times M^d\\
&\tilde{D}(\mathbf{v}_d^{1n};T_d^{1n},\phi_d)=(T_d^{1(n-1)},\phi_d)\quad\forall\phi_d\in W^d\\
&\mathbf{u}_d^{10}=\mathbf{0},\quad T_d^{10}=\psi(x,y),\quad (x,y)\in\Omega\\
&\|\mathbf{u}^{1n}\|_0^2+k\sum_{i=1}^{n}\|\mathbf{u}_d^{1i}\|_1^2\leqslant RM
\end{aligned}\tag{A10}$$

and

$$\begin{aligned}
&B_\delta(\mathbf{v}_d^{2n},\mathbf{v}_d^{2n};\hat{\mathbf{u}}_d^{2n},\hat{\mathbf{w}}_d)=F_\delta(\hat{\mathbf{w}}_d)\quad\forall\hat{\mathbf{w}}_d\in X^d\times M^d\\
&D(\mathbf{v}_d^{2n};T_d^{2n},\phi_d)=(T_d^{2(n-1)},\phi_d)\quad\forall\phi_d\in W^d\\
&\mathbf{u}_d^{20}=\mathbf{0},\quad T_d^{20}=\psi(x,y),\quad (x,y)\in\Omega\\
&\|\mathbf{u}^{2n}\|_0^2+k\sum_{i=1}^{n}\|\mathbf{u}_d^{2i}\|_1^2\leqslant RM
\end{aligned}\tag{A11}$$

By (A10), (A11), (7), (11), (A4), and inverse inequality, we obtain that

$$\begin{aligned}
&\|T_d^{1n}-T_d^{2n}\|_0^2+k\gamma_0^{-1}|T_d^{1n}-T_d^{2n}|_1^2\\
&\quad=(T_d^{1(n-1)}-T_d^{2(n-1)},T_d^{1n}-T_d^{2n})-ka_2(\mathbf{v}_d^{1n}-\mathbf{v}_d^{2n},T_d^{1n},T_d^{1n}-T_d^{2n})\\
&\quad\leqslant\|T_d^{1n}-T_d^{2n}\|_0\|T_d^{1(n-1)}-T_d^{2(n-1)}\|_0+Ck|\mathbf{v}_d^{1n}-\mathbf{v}_d^{2n}|_1|T_d^{1n}-T_d^{2n}|_1\|T_d^{1n}\|_0\\
&\quad\leqslant\tfrac{1}{2}\|T_d^{1n}-T_d^{2n}\|_0^2+\tfrac{1}{2}\|T_d^{1(n-1)}-T_d^{2(n-1)}\|_0^2\\
&\qquad+Ck|\mathbf{v}_d^{1n}-\mathbf{v}_d^{2n}|_1^2+\tfrac{1}{2}k\gamma_0^{-1}|T_d^{1n}-T_d^{2n}|_1^2
\end{aligned}\tag{A12}$$

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Therefore, we obtain

$$\|T_d^{1n}-T_d^{2n}\|_0^2+k\gamma_0^{-1}|T_d^{1n}-T_d^{2n}|_1^2\leqslant\|T_d^{1(n-1)}-T_d^{2(n-1)}\|_0^2+Ck|\mathbf{v}_d^{1n}-\mathbf{v}_d^{2n}|_1^2 \tag{A13}$$

Summing (A13) from 1 to n can yield

$$\|T_d^{1n}-T_d^{2n}\|_0^2+k\gamma_0^{-1}\sum_{i=1}^{n}|T_d^{1i}-T_d^{2i}|_1^2\leqslant Ck\sum_{i=1}^{n}|\mathbf{v}_d^{1i}-\mathbf{v}_d^{2i}|_1^2 \tag{A14}$$

By (A10) and (A11), we obtain, $\forall\hat{\mathbf{w}}_d=(\mathbf{w}_d,r_d)\in X^d\times M^d$,

$$\begin{aligned}
&B_\delta(\mathbf{v}_d^{1n},\mathbf{v}_d^{1n},\hat{\mathbf{u}}_d^{1n},\hat{\mathbf{w}}_d)-B_\delta(\mathbf{v}_d^{2n},\mathbf{v}_d^{2n},\hat{\mathbf{u}}_d^{2n},\hat{\mathbf{w}}_d)\\
&\quad=(\mathbf{u}_d^{1(n-1)}-\mathbf{u}_d^{2(n-1)},\mathbf{w}_d)+\sum_{K\in\Im_h}\delta_K(\mathbf{u}_d^{2(n-1)},k(\mathbf{v}_d^{1n}-\mathbf{v}_d^{2n})\cdot\nabla\mathbf{w}_d)_K\\
&\qquad+\sum_{K\in\Im_h}\delta_K(\mathbf{u}_d^{1(n-1)}-\mathbf{u}_d^{2(n-1)},\mathbf{w}_d-kv\Delta\mathbf{w}_d+k\mathbf{v}_d^{1n}\cdot\nabla\mathbf{w}_d+k\nabla r_d)_K\\
&\qquad+\sum_{K\in\Im_h}\delta_K(kj(T_d^{1n}-T_d^{2n}),\mathbf{w}_d-kv\Delta\mathbf{w}_d+k\mathbf{v}_d^{1n}\cdot\nabla\mathbf{w}_d+k\nabla r_d)_K\\
&\qquad+k(j(T_d^{1n}-T_d^{2n}),\mathbf{w}_d)+\sum_{K\in\Im_d}\delta_K(kjT_d^{2n},k(\mathbf{v}_d^{1n}-\mathbf{v}_d^{2n})\nabla\mathbf{w}_d)\equiv S_0
\end{aligned} \tag{A15}$$

Taking $\mathbf{w}_d=\mathbf{u}_d^{1n}-\mathbf{u}_d^{2n}$ and $r_d=p_d^{1n}-p_d^{2n}$, on the one hand, we obtain

$$\begin{aligned}
B_\delta(\mathbf{v}_d^{1n},\mathbf{v}_d^{1n},\hat{\mathbf{w}}_d,\hat{\mathbf{w}}_d)&=\|\mathbf{w}_d\|_0^2+kv|\mathbf{w}_d|_1^2\\
&\quad+\|\delta^{1/2}(\mathbf{w}_d-kv\Delta\mathbf{w}_d+k\mathbf{v}_d^{1n}\cdot\nabla\mathbf{w}_d+k\nabla r_d)\|_{0,h}^2
\end{aligned} \tag{A16}$$

On the other hand, by (A10) and (A11) we obtain

$$\begin{aligned}
B_\delta(\mathbf{v}_d^{1n},\mathbf{v}_d^{1n},\hat{\mathbf{w}}_d,\hat{\mathbf{w}}_d)&=B_\delta(\mathbf{v}_d^{1n},\mathbf{v}_d^{1n},\hat{\mathbf{u}}_d^{1n},\hat{\mathbf{w}}_d)-B_\delta(\mathbf{v}_d^{1n},\mathbf{v}_d^{1n},\hat{\mathbf{u}}_d^{2n},\hat{\mathbf{w}}_d)\\
&=B_\delta(\mathbf{v}_d^{2n},\mathbf{v}_d^{2n},\hat{\mathbf{u}}_d^{2n},\hat{\mathbf{w}}_d)-B_\delta(\mathbf{v}_d^{1n},\mathbf{v}_d^{1n},\hat{\mathbf{u}}_d^{2n},\hat{\mathbf{w}}_d)+S_0\\
&=ka_1(\mathbf{v}_d^{2n}-\mathbf{v}_d^{1n},\mathbf{u}_d^{2n},\mathbf{w}_d)\\
&\quad+\sum_{K\in\Im_h}\delta_K(k(\mathbf{v}_d^{2n}-\mathbf{v}_d^{1n})\nabla\mathbf{u}_d^{2n},\mathbf{w}_d-kv\Delta\mathbf{w}_d+k\mathbf{v}_d^{1n}\cdot\nabla\mathbf{w}_d+k\nabla r_d)_K\\
&\quad+\sum_{K\in\Im_h}\delta_K(\mathbf{u}_d^{2n}-kv\Delta\mathbf{u}_d^{2n}+k\mathbf{v}_d^{2n}\cdot\nabla\mathbf{u}_d^{2n}+k\nabla p_d^{2n},k(\mathbf{v}_d^{2n}-\mathbf{v}_d^{1n})\nabla\mathbf{w}_d)_K+S_0\\
&\equiv S_1+S_2+S_3+S_0
\end{aligned} \tag{A17}$$

By (11) and (A11), we obtain

$$|S_1|=|ka_1(\mathbf{v}_d^{2n}-\mathbf{v}_d^{1n},\mathbf{u}_d^{2n},\mathbf{w}_d)|\leqslant kRMN_0|\mathbf{v}_d^{2n}-\mathbf{v}_d^{1n}|_1|\mathbf{w}_d|_1 \tag{A18}$$

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By inverse inequality (see [36, 37]), (11), (A11), and (A12), we obtain

$$
\begin{aligned}
|S_2| &= \left|\sum_{K\in\Im_h}\delta_K(k(\mathbf{v}_d^{2n}-\mathbf{v}_d^{1n})\nabla\mathbf{u}_d^{2n}, \mathbf{w}_d - k\nu\Delta\mathbf{w}_d + k\mathbf{v}_d^{1n}\cdot\nabla\mathbf{w}_d + k\nabla r_d)_K\right| \\
&\leqslant Ckh^{1/2}\|\mathbf{v}_d^{2n}-\mathbf{v}_d^{1n}\|_1\|\delta^{1/2}(\mathbf{w}_d - k\nu\Delta\mathbf{w}_d + k\mathbf{v}_d^{1n}\cdot\nabla\mathbf{w}_d + k\nabla r_d)\|_{0,h} \qquad \text{(A19)}
\end{aligned}
$$

$$
\begin{aligned}
|S_3| &= \left|\sum_{K\in\Im_h}\delta_K(\mathbf{u}_d^{2n} - k\nu\Delta\mathbf{u}_d^{2n} + k\mathbf{v}_d^{2n}\cdot\nabla\mathbf{u}_d^{2n} + k\nabla p_d^{2n}, k(\mathbf{v}_d^{2n}-\mathbf{v}_d^{1n})\nabla\mathbf{w}_d)_K\right| \\
&\leqslant Ckh^{1/2}|\mathbf{v}_d^{2n}-\mathbf{v}_d^{1n}|_1|\mathbf{w}_d|_1 \qquad \text{(A20)}
\end{aligned}
$$

$$
\begin{aligned}
|S_0| &= \Bigg|(\mathbf{u}_d^{1(n-1)}-\mathbf{u}_d^{2(n-1)}, \mathbf{w}_d) + \sum_{K\in\Im_h}\delta_K(\mathbf{u}_d^{2(n-1)}, k(\mathbf{v}_d^{1n}-\mathbf{v}_d^{2n})\cdot\nabla\mathbf{w}_d)_K \\
&\quad + \sum_{K\in\Im_h}\delta_K(\mathbf{u}_d^{1(n-1)}-\mathbf{u}_d^{2(n-1)}, \mathbf{w}_d - k\nu\Delta\mathbf{w}_d + k\mathbf{v}_d^{1n}\cdot\nabla\mathbf{w}_d + k\nabla r_d)_K \\
&\quad + \sum_{K\in\Im_h}\delta_K(kj(T_d^{1n}-T_d^{2n}), \mathbf{w}_d - k\nu\Delta\mathbf{w}_d + k\mathbf{v}_d^{1n}\cdot\nabla\mathbf{w}_d + k\nabla r_d)_K \\
&\quad + k(j(T_d^{1n}-T_d^{2n}), \mathbf{w}_d) + \sum_{K\in\Im_h}\delta_K(jT_d^{2n}, k(\mathbf{v}_d^{1n}-\mathbf{v}_d^{2n})\nabla\mathbf{w}_d)\Bigg| \\
&\leqslant \|\mathbf{u}_d^{1(n-1)}-\mathbf{u}_d^{2(n-1)}\|_0\|\mathbf{w}_d\|_0 + Ckh|\mathbf{v}_d^{1n}-\mathbf{v}_d^{2n}|_1|\mathbf{w}_d|_1 \\
&\quad + Ch^{1/2}\|\mathbf{u}_d^{1(n-1)}-\mathbf{u}_d^{2(n-1)}\|_0\|\delta^{1/2}(\mathbf{w}_d - k\nu\Delta\mathbf{w}_d + k\mathbf{v}_d^{1n}\cdot\nabla\mathbf{w}_d + k\nabla r_d)\|_{0,h} \\
&\quad + Ckh^{1/2}\|T_d^{1n}-T_d^{2n}\|_0\|\delta^{1/2}(\mathbf{w}_d - k\nu\Delta\mathbf{w}_d + k\mathbf{v}_d^{1n}\cdot\nabla\mathbf{w}_d + k\nabla r_d)\|_{0,h} \\
&\quad + kh\|T_d^{1n}-T_d^{2n}\|_0\|\mathbf{w}_d\|_1 + Chk|\mathbf{v}_d^{1n}-\mathbf{v}_d^{2n}|_1|\mathbf{w}_d|_1 \qquad \text{(A21)}
\end{aligned}
$$

If $h = O(k)$, combining (A18)–(A21) and (A14) and using the Cauchy inequality could yield

$$
\begin{aligned}
|S_0|+|S_1|+|S_2|+|S_3| &\leqslant \frac{1}{2}\|\mathbf{u}_d^{1(n-1)}-\mathbf{u}_d^{2(n-1)}\|_0^2 + \frac{1}{2}\|\mathbf{w}_d\|_0 \\
&\quad + Ck^2|\mathbf{v}_d^{1n}-\mathbf{v}_d^{2n}|_1^2 + \frac{k\nu}{2}|\mathbf{w}_d|_1^2 + Ck\|\mathbf{u}_d^{1(n-1)}-\mathbf{u}_d^{2(n-1)}\|_0^2 \\
&\quad + \frac{1}{2}\|\delta^{1/2}(\mathbf{w}_d - k\nu\Delta\mathbf{w}_d + k\mathbf{v}_d^{1n}\cdot\nabla\mathbf{w}_d + k\nabla r_d)\|_{0,h}^2 \\
&\quad + Ck^3\sum_{i=1}^{n}|\mathbf{v}_d^{1i}-\mathbf{v}_d^{2i}|_1^2 \qquad \text{(A22)}
\end{aligned}
$$

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Combining (A22) and (A16), (A17) yields

$$\begin{aligned}&\|\delta^{1/2}(\mathbf{u}_d^{1n}-\mathbf{u}_d^{2n})-k\nu\Delta(\mathbf{u}_d^{1n}-\mathbf{u}_d^{2n})+k\mathbf{v}_d^{1n}\cdot\nabla(\mathbf{u}_d^{1n}-\mathbf{u}_d^{2n})\\&\quad+k\nabla(p_d^{1n}-p_d^{2n})\|_{0,h}^2+\|\mathbf{u}_d^{2n}-\mathbf{u}_d^{1n}\|_0^2+k\nu|\mathbf{u}_d^{2n}-\mathbf{u}_d^{1n}|_1^2\\&\leqslant\|\mathbf{u}_d^{1(n-1)}-\mathbf{u}_d^{2(n-1)}\|_0^2+Ck\|\mathbf{u}_d^{1(n-1)}-\mathbf{u}_d^{2(n-1)}\|_0^2+Ck^3\sum_{i=1}^n|\mathbf{v}_d^{1i}-\mathbf{v}_d^{2i}|_1^2\end{aligned}\tag{A23}$$

Summing (A23) from 1 to n we obtain

$$\begin{aligned}&\sum_{i=1}^n\|\delta^{1/2}(\mathbf{u}_d^{1i}-\mathbf{u}_d^{2i})-k\nu\Delta(\mathbf{u}_d^{1i}-\mathbf{u}_d^{2i})+k\mathbf{v}_d^{1i}\cdot\nabla(\mathbf{u}_d^{1i}-\mathbf{u}_d^{2i})\\&\quad+k\nabla(p_d^{1i}-p_d^{2i})\|_{0,h}^2+\|\mathbf{u}_d^{1n}-\mathbf{u}_d^{2n}\|_0^2+k\nu\sum_{i=1}^n|\mathbf{u}_d^{1i}-\mathbf{u}_d^{2i}|_1^2\\&\leqslant Ck\sum_{i=0}^{n-1}\|\mathbf{u}_d^{1i}-\mathbf{u}_d^{2i}\|_0^2+Ck^3n\sum_{i=1}^n|\mathbf{v}_d^{1i}-\mathbf{v}_d^{2i}|_1^2\end{aligned}\tag{A24}$$

By discrete Gronwall inequality, we obtain

$$\begin{aligned}&\sum_{i=1}^n\|\delta^{1/2}(\mathbf{u}_d^{1i}-\mathbf{u}_d^{2i})-k\nu\Delta(\mathbf{u}_d^{1i}-\mathbf{u}_d^{2i})+k\mathbf{v}_d^{1i}\cdot\nabla(\mathbf{u}_d^{1i}-\mathbf{u}_d^{2i})+k\nabla(p_d^{1i}-p_d^{2i})\|_{0,h}^2\\&\quad+\|\mathbf{u}_d^{1n}-\mathbf{u}_d^{2n}\|_0^2+k\nu\sum_{i=1}^n|\mathbf{u}_d^{1i}-\mathbf{u}_d^{2i}|_1^2\leqslant Ck^3n\sum_{i=1}^n|\mathbf{v}_d^{1i}-\mathbf{v}_d^{2i}|_1^2\exp(Cnk)\end{aligned}\tag{A25}$$

Since $nk\leqslant t_N$, we obtain

$$\begin{aligned}&\sum_{i=1}^n\|\delta^{1/2}(\mathbf{u}_d^{1i}-\mathbf{u}_d^{2i})-k\nu\Delta(\mathbf{u}_d^{1i}-\mathbf{u}_d^{2i})+k\mathbf{v}_d^{1i}\cdot\nabla(\mathbf{u}_d^{1i}-\mathbf{u}_d^{2i})+k\nabla(p_d^{1i}-p_d^{2i})\|_{0,h}^2\\&\quad+\|\mathbf{u}_d^{1n}-\mathbf{u}_d^{2n}\|_0^2+k\nu\sum_{i=1}^n|\mathbf{u}_d^{1i}-\mathbf{u}_d^{2i}|_1^2\leqslant Ck^2\sum_{i=1}^n|\mathbf{v}_d^{1i}-\mathbf{v}_d^{2i}|_1^2\end{aligned}\tag{A26}$$

Thus, (A14) and (A26) show that the map $G:B_{RM}\to B_{RM}$ is continuous. By Brouwer's fixed point theorem, this implies that G has at least one fixed $(\hat{\mathbf{u}}_d^n,T_d^n)=G(\hat{\mathbf{u}}_d^n,T_d^n)$ $(n=1,2,\ldots,N)$, i.e. Problem (IV) has at least one solution sequence $(\mathbf{u}_d^n,p_d^n,T_d^n)\in X^d\times M^d\times W^d$.

If $(\mathbf{u}_d^{1n},p_d^{1n},T_d^{1n})\in X^d\times M^d\times W^d$ and $(\mathbf{u}_d^{21n},p_d^{2n},T_d^{2n})\in X^d\times M^d\times W^d$ are two groups of solutions for Problem (V), using the same approach as in (A14) and (A26), we derive

$$\|T_d^{1n}-T_d^{2n}\|_0^2+k\gamma_0^{-1}\sum_{i=1}^n|T_d^{1i}-T_d^{2i}|_1^2\leqslant Ck\sum_{i=1}^n|\mathbf{u}_d^{1i}-\mathbf{u}_d^{2i}|_1^2\tag{A27}$$

$$\begin{aligned}&\sum_{i=1}^n\|\delta^{1/2}(\mathbf{u}_d^{1i}-\mathbf{u}_d^{2i})-k\nu\Delta(\mathbf{u}_d^{1i}-\mathbf{u}_d^{2i})+k\mathbf{u}_d^{1i}\cdot\nabla(\mathbf{u}_d^{1i}-\mathbf{u}_d^{2i})+k\nabla(p_d^{1i}-p_d^{2i})\|_{0,h}^2\\&\quad+\|\mathbf{u}_d^{1n}-\mathbf{u}_d^{2n}\|_0^2+k\nu\sum_{i=1}^n|\mathbf{u}_d^{1i}-\mathbf{u}_d^{2i}|_1^2\leqslant Ckh\sum_{i=1}^n|\mathbf{u}_d^{1i}-\mathbf{u}_d^{2i}|_1^2\end{aligned}\tag{A28}$$

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Therefore, there is an $h_0 = \nu/(2C)$ such that if $h \leqslant h_0$, we obtain

$$\sum_{i=1}^{n} \|\delta^{1/2}(\mathbf{u}_d^{1i} - \mathbf{u}_d^{2i}) - k\nu\Delta(\mathbf{u}_d^{1i} - \mathbf{u}_d^{2i}) + k\mathbf{u}_d^{1i} \cdot \nabla(\mathbf{u}_d^{1i} - \mathbf{u}_d^{2i}) + k\nabla(p_d^{1i} - p_d^{2i})\|_{0,h}^2$$

$$+\|\mathbf{u}_d^{1n} - \mathbf{u}_d^{2n}\|_0^2 + k\nu \sum_{i=1}^{n} |\mathbf{u}_d^{1i} - \mathbf{u}_d^{2i}|_1^2 \leqslant 0 \tag{A29}$$

which shows that $\mathbf{u}_d^{1n} = \mathbf{u}_d^{2n}$ and $p_d^{1n} = p_d^{2n}$. And by (A27) we obtain $T_d^{1n} = T_d^{2n}$. Therefore, the solutions $(\mathbf{u}_d^n, p_d^n, T_d^n)$ $(1 \leqslant n \leqslant N)$ for Problem (V) are unique. □

APPENDIX B

The proof of Theorem 4.2 is as follows.

Let $\hat{\mathbf{w}}_d = (\mathbf{w}_d^n, r_d^n)$, $\mathbf{w}_d^n = P^d\mathbf{u}_h^n - \mathbf{u}_d^n$, and $r_d^n = \rho^d p_h^n - p_d^n$. On the one hand, we have

$$B_\delta(\mathbf{u}_d^n, \mathbf{u}_d^n, \hat{\mathbf{w}}_d, \hat{\mathbf{w}}_d) = \|\mathbf{w}_d^n\|_0^2 + k\nu|\mathbf{w}_d^n|_1^2 + \|\delta^{1/2}(\mathbf{w}_d^n - k\nu\Delta\mathbf{w}_d^n + k\mathbf{u}_d^n\nabla\mathbf{w}_d^n + k\nabla r_d^n)\|_{0,h}^2 \tag{B1}$$

On the other hand, if write $\hat{\mathbf{P}}^d\hat{\mathbf{u}} = (P^d\mathbf{u}_h^n, \rho^d p_h^n)$ and $\hat{\mathbf{u}}_d = (\mathbf{u}_d^n, p_h^d)$, we have

$$\begin{aligned}
B_\delta(\mathbf{u}_d^n, \mathbf{u}_d^n, \hat{\mathbf{w}}_d, \hat{\mathbf{w}}_d) &= B_\delta(\mathbf{u}_d^n, \mathbf{u}_d^n, \hat{P}^d\hat{\mathbf{u}}, \hat{\mathbf{w}}_d) - B_\delta(\mathbf{u}_d^n, \mathbf{u}_d^n, \hat{\mathbf{u}}_d, \hat{\mathbf{w}}_d) \\
&= B_\delta(\mathbf{u}_d^n, \mathbf{u}_d^n, \hat{P}^d\hat{\mathbf{u}}, \hat{\mathbf{w}}_d) - B_\delta(\mathbf{u}_h^n, \mathbf{u}_d^n, \hat{\mathbf{u}}_h^n, \hat{\mathbf{w}}_d) + (\mathbf{u}_h^{n-1} - \mathbf{u}_d^{n-1}, \mathbf{w}_d^n) \\
&\quad + \sum_{K\in\Im_h} \delta_K(\mathbf{u}_h^{n-1} - \mathbf{u}_d^{n-1} + kj(T_h^n - T_d^n), \mathbf{w}_d^n - k\nu\Delta\mathbf{w}_d^n + k\mathbf{u}_d^n\nabla\mathbf{w}_d^n + k\nabla r_d^n)_K \\
&\quad + kj(T_h^n - T_d^n, \mathbf{w}_d^n) \\
&\equiv \bar{S}_1 + \bar{S}_2 + \bar{S}_3 + \bar{S}_4
\end{aligned} \tag{B2}$$

where, since $a(P^d\mathbf{u}_h^n - \mathbf{u}_h^n, \mathbf{w}_d^n) = 0$,

$$\bar{S}_1 = (P^d\mathbf{u}_h^n - \mathbf{u}_h^n, \mathbf{w}_d^n) - kb(\rho^d p_h^n - p_h^n, \mathbf{w}_d^n)$$

$$\bar{S}_2 = k[a_1(\mathbf{u}_d^n, P^d\mathbf{u}_h^n, \mathbf{w}_d^n) - a_1(\mathbf{u}_h^n, \mathbf{u}_h^n, \mathbf{w}_d^n) + b(r_d^n, P^d\mathbf{u}_h^n - \mathbf{u}_h^n)]$$

$$\begin{aligned}
\bar{S}_3 = \sum_{K\in\Im_h} \delta_K(&P^d\mathbf{u}_h^n - \mathbf{u}_h^n - k\nu\Delta(P^d\mathbf{u}_h^n - \mathbf{u}_h^n) + k\mathbf{u}_d^n\nabla P^d\mathbf{u}_h^n - k\mathbf{u}_h^n\nabla\mathbf{u}_h^n \\
&+ k\nabla(\rho^d p_h^n - p_h^n), \mathbf{w}_d^n - k\nu\Delta\mathbf{w}_d^n + k\mathbf{u}_d^n\nabla\mathbf{w}_d^n + k\nabla r_d^n)_K
\end{aligned}$$

$$\begin{aligned}
\bar{S}_4 = (\mathbf{u}_h^{n-1} - \mathbf{u}_d^{n-1}, \mathbf{w}_d^n) + kj(T_h^n - T_d^n, \mathbf{w}_d^n) + \sum_{K\in\Im_h} \delta_K(&\mathbf{u}_h^{n-1} - \mathbf{u}_d^{n-1} \\
&+ kj(T_h^n - T_d^n), \mathbf{w}_d^n - k\nu\Delta\mathbf{w}_d^n + k\mathbf{u}_d^n\nabla\mathbf{w}_d^n + k\nabla r_d^n)_K
\end{aligned}$$

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Using the inverse inequality, Hölder inequality, (8), and Cauchy inequality and noting that $h=O(k)$, we obtain

$$|\bar{S}_1| = |(P^d\mathbf{u}_h^n-\mathbf{u}_h^n,\mathbf{w}_d^n)-kb(\rho^d p_h^n-p_h^n,\mathbf{w}_d^n)| \leqslant Ck(|P^d\mathbf{u}_h^n-\mathbf{u}_h^n|_1^2+\|\rho^d p_h^n-p_h^n\|_0^2)+\tilde{\varepsilon}k\nu|\mathbf{w}_d^n|_1^2 \tag{B3}$$

$$\begin{aligned}|\bar{S}_2| &= k|a_1(\mathbf{u}_d^n,P^d\mathbf{u}_h^n,\mathbf{w}_d^n)-a_1(\mathbf{u}_h^n,\mathbf{u}_h^n,\mathbf{w}_d^n)+b(r_d^n,P^d\mathbf{u}_h^n-\mathbf{u}_h^n)|\\ &= k\Bigg|a_1(\mathbf{u}_d^n,P^d\mathbf{u}_h^n-\mathbf{u}_h^n,\mathbf{w}_d^n)-a_1(\mathbf{w}_d^n,\mathbf{u}_h^n,\mathbf{w}_d^n)+a_1(P^d\mathbf{u}_h^n-\mathbf{u}_h^n,\mathbf{u}_d^n,\mathbf{w}_d^n)\\ &\quad -\sum_{K\in\Im_h}(\mathbf{w}_d^n-k\nu\Delta\mathbf{w}_d^n+k\mathbf{u}_d^n\cdot\mathbf{w}_d^n+\nabla r_d^n,P^d\mathbf{u}_h^n-\mathbf{u}_h^n)_K\\ &\quad +\sum_{K\in\Im_h}(\mathbf{w}_d^n-k\nu\Delta\mathbf{w}_d^n+k\mathbf{u}_d^n\cdot\mathbf{w}_d^n,P^d\mathbf{u}_h^n-\mathbf{u}_h^n)_K\Bigg|\\ &\leqslant \tilde{\varepsilon}(k\nu|\mathbf{w}_d^n|_1^2+\|\delta^{1/2}(\mathbf{w}_d^n-k\nu\Delta\mathbf{w}_d^n+k\mathbf{u}_d^n\cdot\mathbf{w}_d^n+k\nabla r_d^n)\|_{0,h}^2)\\ &\quad +Ck|P^d\mathbf{u}_h^n-\mathbf{u}_h^n|_1^2+Ckh|\mathbf{w}_d^n|_1^2\end{aligned} \tag{B4}$$

$$\begin{aligned}|\bar{S}_3| &= \sum_{K\in\Im_h}\delta_K(P^d\mathbf{u}_h^n-\mathbf{u}_h^n-k\nu\Delta(P^d\mathbf{u}_h^n-\mathbf{u}_h^n)+k\mathbf{u}_d^n\nabla(P^d\mathbf{u}_h^n-\mathbf{u}_h^n)-k\mathbf{w}_d^n\nabla\mathbf{u}_h^n\\ &\quad +k(P^d\mathbf{u}_h^n-\mathbf{u}_h^n)\nabla\mathbf{u}_h^n+k\nabla(\rho^d p_h^n-p_h^n),\mathbf{w}_d^n-k\nu\Delta\mathbf{w}_d^n+k\mathbf{u}_d^n\nabla\mathbf{w}_d^n+k\nabla r_d^n)_K\\ &\leqslant Ck(|P^d\mathbf{u}_h^n-\mathbf{u}_h^n|_1^2+\|\rho^d p_h^n-p_h^n\|_0^2)+Ckh|\mathbf{w}_d^n|_1^2\\ &\quad +\tilde{\varepsilon}\|\delta^{1/2}(\mathbf{w}_d^n-k\nu\Delta\mathbf{w}_d^n+k\mathbf{u}_d^n\nabla\mathbf{w}_d^n+k\nabla r_d^n)\|_{0,h}^2\end{aligned} \tag{B5}$$

$$\begin{aligned}|\bar{S}_4| &\leqslant Ck|P^d\mathbf{u}_h^{n-1}-\mathbf{u}_h^{n-1}|_1^2+\tfrac{1}{2}(\|\mathbf{w}_d^{n-1}\|_0^2+\|\mathbf{w}_d^n\|_0^2)+Ck\|\mathbf{w}_d^{n-1}\|_0^2\\ &\quad +\tilde{\varepsilon}\|\delta^{1/2}(\mathbf{w}_d^n-k\nu\Delta\mathbf{w}_d^n+k\mathbf{u}_d^n\nabla\mathbf{w}_d^n+k\nabla r_d^n)\|_{0,h}^2\\ &\quad +\tilde{\varepsilon}k\nu|\mathbf{w}_d^n|_1+Ck^3\|T_h^n-T_d^n\|_0^2\end{aligned} \tag{B6}$$

where $\tilde{\varepsilon}$ is a constant, which can be chosen arbitrarily. Combining (B1) and (B2)–(B6) could yield

$$\begin{aligned}&\|\mathbf{w}_d^n\|_0^2+k\nu|\mathbf{w}_d^n|_1^2+\|\delta^{1/2}(\mathbf{w}_d^n-k\nu\Delta\mathbf{w}_d^n+k\mathbf{u}_d^n\nabla\mathbf{w}_d^n+k\nabla r_d^n)\|_{0,h}^2\\ &\quad\leqslant Ck(|P^d\mathbf{u}_h^n-\mathbf{u}_h^n|_1^2+\|\rho^d p_h^n-p_h^n\|_0^2+|P^d\mathbf{u}_h^{n-1}-\mathbf{u}_h^{n-1}|_1^2)\\ &\qquad +3\tilde{\varepsilon}k(\nu|\mathbf{w}_d^n|_1^2+\|\delta^{1/2}(\mathbf{w}_d^n-k\nu\Delta\mathbf{w}_d^n+k\mathbf{u}_d^n\nabla\mathbf{w}_d^n+k\nabla r_d^n)\|_{0,h}^2)\\ &\qquad +Ckh|\mathbf{w}_d^n|_1^2+\tfrac{1}{2}(\|\mathbf{w}_d^{n-1}\|_0^2+\|\mathbf{w}_d^n\|_0^2)+Ck\|\mathbf{w}_d^{n-1}\|_0^2+Ck^3\|T_h^n-T_d^n\|_0^2\end{aligned} \tag{B7}$$

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Taking $\tilde{\varepsilon} \leqslant \frac{1}{6}$, from (B7) we obtain

$$\begin{aligned}&\|\mathbf{w}_d^n\|_0^2 + k\nu|\mathbf{w}_d^n|_1^2 + \|\delta^{1/2}(\mathbf{w}_d^n - k\nu\Delta\mathbf{w}_d^n + k\mathbf{u}_d^n\nabla\mathbf{w}_d^n + k\nabla r_d^n)\|_{0,h}^2\\ &\quad\leqslant Ck(|P^d\mathbf{u}_h^n - \mathbf{u}_h^n|_1^2 + \|\rho^d p_h^n - p_h^n\|_0^2 + |P^d\mathbf{u}_h^{n-1} - \mathbf{u}_h^{n-1}|_1^2)\\ &\qquad + Ckh|\mathbf{w}_d^n|_1^2 + \|\mathbf{w}_d^{n-1}\|_0^2 + Ck\|\mathbf{w}_d^{n-1}\|_0^2 + Ck^3\|T_h^n - T_d^n\|_0^2 \end{aligned} \tag{B8}$$

If h is sufficiently small such that $Ch \leqslant \nu/2$, we could obtain from the above inequality that

$$\begin{aligned}&2\|\mathbf{w}_d^n\|_0^2 + k\nu|\mathbf{w}_d^n|_1^2 + 2\|\delta^{1/2}(\mathbf{w}_d^n - k\nu\Delta\mathbf{w}_d^n + k\mathbf{u}_d^n\nabla\mathbf{w}_d^n + k\nabla r_d^n)\|_{0,h}^2\\ &\quad\leqslant Ck(|P^d\mathbf{u}_h^n - \mathbf{u}_h^n|_1^2 + \|\rho^d p_h^n - p_h^n\|_0^2 + |P^d\mathbf{u}_h^{n-1} - \mathbf{u}_h^{n-1}|_1^2)\\ &\qquad + 2\|\mathbf{w}_d^{n-1}\|_0^2 + Ck\|\mathbf{w}_d^{n-1}\|_0^2 + Ck^3\|T_h^n - T_d^n\|_0^2 \end{aligned} \tag{B9}$$

Let $\tau_d^n = \varrho^d T_h^n - T_d^n$. By (7), (A10), (A11), and inverse inequality, we could get that

$$\begin{aligned}\|\tau_d^n\|_0^2 + k\gamma_0^{-1}|\tau_d^n|_1^2 &= (\varrho^d T_h^n - T_h^n, \tau_d^n) + (\tau_d^{n-1}, \tau_d^n) - ka_2(\mathbf{u}_h^n - \mathbf{u}_d^n, T_h^n, \tau_d^n)\\ &\quad + ka_2(\mathbf{u}_d^n, \varrho^d T_h^n - T_h^n, \tau_d^n) + (T_h^{n-1} - \varrho^d T_h^{n-1}, \tau_d^n)\\ &\leqslant \frac{k\gamma_0^{-1}}{2}|\tau_d^n|_1^2 + \frac{1}{2}\|\tau_d^n\|_0^2 + \frac{1}{2}\|\tau_d^{n-1}\|_0^2 + Ck|\mathbf{u}_h^n - \mathbf{u}_d^n|_1^2\\ &\quad + Ck\|T_h^{n-1} - \varrho^d T_h^{n-1}\|_0^2 + Ck|T_h^n - \varrho^d T_h^n|_1^2 \end{aligned} \tag{B10}$$

Therefore, we have

$$\begin{aligned}\|\tau_d^n\|_0^2 + k\gamma_0^{-1}|\tau_d^n|_1^2 &\leqslant \|\tau_d^{n-1}\|_0^2 + Ck\|T_h^{n-1} - \varrho^d T_h^{n-1}\|_0^2\\ &\quad + Ck|T_h^n - \varrho^d T_h^n|_1^2 + Ck|\mathbf{u}_h^n - \mathbf{u}_d^n|_1^2 \end{aligned} \tag{B11}$$

First, we consider the case of $n \in \{n_1, n_2, \ldots, n_L\}$. Summing (B11) from $n = n_1$ to $n_i \in \{n_1, n_2, \ldots, n_L\}$ and using Lemma 3.2 could yield

$$\|\tau_d^{n_i}\|_0^2 + k\gamma_0^{-1}\sum_{j=n_1}^{n_i}|\tau_d^j|_1^2 \leqslant Ck\sum_{j=n_1}^{n_i}|\mathbf{u}_h^j - \mathbf{u}_d^j|_1^2 + CkL\sum_{j=d+1}^{l}\lambda_j \tag{B12}$$

Thus, if $k = O(L^{-2})$,

$$\|T^{n_i} - T_d^{n_i}\|_0^2 + k\gamma_0^{-1}\sum_{j=n_1}^{n_i}|T^j - T_d^j|_1^2 \leqslant Ck\sum_{j=n_1}^{n_i}|\mathbf{u}_h^j - \mathbf{u}_d^j|_1^2 + Ck^{1/2}\sum_{j=d+1}^{l}\lambda_j \tag{B13}$$

Summing (B9) from n_1 to $n_i \in \{n_1, n_2, \ldots, n_L\}$ and using Lemma 3.2 yield

$$\begin{aligned}&\|\mathbf{w}_d^{n_i}\|_0^2 + k\nu\sum_{j=n_1}^{n_i}|\mathbf{w}_d^j|_1^2 + \sum_{j=n_1}^{n_i}\|\delta^{1/2}(\mathbf{w}_d^j - k\nu\Delta\mathbf{w}_d^j + k\mathbf{u}_d^j\nabla\mathbf{w}_d^j + k\nabla r_d^j)\|_{0,h}^2\\ &\quad\leqslant CkL\sum_{j=d+1}^{l}\lambda_j + Ck\sum_{j=n_0}^{n_{i-1}}\|\mathbf{w}_d^j\|_0^2 + Ck^3\sum_{j=n_1}^{n_i}\|T^j - T_d^j\|_0^2 \end{aligned} \tag{B14}$$

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By using the discrete Gronwall inequality, we obtain

$$\|\mathbf{w}_d^{n_i}\|_0^2+k\nu\sum_{j=n_1}^{n_i}|\mathbf{w}_d^j|_1^2+\sum_{j=n_1}^{n_i}\|\delta^{1/2}(\mathbf{w}_d^j-k\nu\Delta\mathbf{w}_d^j+k\mathbf{u}_d^j\nabla\mathbf{w}_d^j+k\nabla r_d^j)\|_{0,h}^2$$
$$\leqslant Ck\left[L\sum_{j=d+1}^{l}\lambda_j+k^2\sum_{j=n_1}^{n_i}\|T^j-T_d^j\|_0^2\right]\exp(Cki) \tag{B15}$$

If h and k are sufficiently small, $k=O(L^{-2})$, by using inverse inequality and noting that $ik\leqslant kN\leqslant T$, we obtain

$$\|\mathbf{w}_d^{n_i}\|_0+(k\nu)^{1/2}\sum_{j=n_1}^{n_i}|\mathbf{w}_d^j|_1+k^{1/2}\sum_{j=n_1}^{n_i}\|r_d^j\|_0$$
$$\leqslant C\left(k^{1/2}\sum_{j=d+1}^{l}\lambda_j\right)^{1/2}+C\left[k^3\sum_{j=n_1}^{n_i}\|T^j-T_d^j\|_0^2\right]^{1/2} \tag{B16}$$

Using Lemma 3.2 and (B13) yields

$$\|\mathbf{u}_h^{n_i}-\mathbf{u}_d^{n_i}\|_0+(k\nu)^{1/2}\sum_{j=n_1}^{n_i}|\mathbf{u}_h^j-\mathbf{u}_d^j|_1+k^{1/2}\sum_{j=n_1}^{n}\|p_h^j-p_d^j\|_0$$
$$\leqslant C\left(k^{1/2}\sum_{j=d+1}^{l}\lambda_j\right)^{1/2}+Ck^{3/2}\sum_{j=n_1}^{n_i}|\mathbf{u}_h^j-\mathbf{u}_d^j|_1 \tag{B17}$$

If k is sufficiently small, for example, $Ck\leqslant(\nu)^{1/2}/2$, by (B17) we obtain

$$\|\mathbf{u}_h^{n_i}-\mathbf{u}_d^{n_i}\|_0+(k\nu)^{1/2}\sum_{j=n_1}^{n_i}|\mathbf{u}_h^j-\mathbf{u}_d^j|_1+k^{1/2}\sum_{j=n_1}^{n_i}\|p_h^j-p_d^j\|_0\leqslant C\left(k^{1/2}\sum_{j=d+1}^{l}\lambda_j\right)^{1/2} \tag{B18}$$

Combining (B18) and (B13) can yield (28).

Next, we consider the case of $n\notin\{n_1,n_2,\ldots,n_L\}$. If $n\notin\{n_1,n_2,\ldots,n_L\}$; we may as well suppose that $t_n\in(t_{n_i},t_{n_{i+1}})$. Expanding $\mathbf{u}_h^n$, p_h^n, and T_h^n into Taylor series with respect to t_{n_i} could yield

$$\mathbf{u}_h^n=\mathbf{u}_h^{n_i}-\eta k\frac{\partial\mathbf{u}_h(\xi_1)}{\partial t},\quad \xi_1\in[t_i,\ t_n]$$
$$p_h^n=p_h^{n_i}-\eta k\frac{\partial p_h(\xi_2)}{\partial t},\quad \xi_2\in[t_i,\ t_n] \tag{B19}$$
$$T_h^n=T_h^{n_i}-\eta k\frac{\partial T_h(\xi_3)}{\partial t},\quad \xi_3\in[t_i,\ t_n]$$

where η is the step number from t_{n_i} to t_n. If snapshots are equably taken, then $\eta\leqslant N/L$. Summing (B11) and (B9) from n_1 to n_i, n, and using (B19), if $|\partial\mathbf{u}_h(\xi_1)/\partial t|$, $|\partial p_h(\xi_2)/\partial t|$, and $|\partial T_h(\xi_3)/\partial t|$

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are bounded, by discrete Gronwall inequality and Lemma 3.2, we obtain

$$\|\tau_d^n\|_0^2+k\gamma_0^{-1}\left[|\tau_d^n|_1^2+\sum_{j=n_1}^{n_i}|\tau_d^i|_1^2\right]$$

$$\leqslant Ck\sum_{j=n_1}^{n_i}|\mathbf{u}_h^j-\mathbf{u}_d^j|_1^2+CkL\sum_{j=d+1}^{l}\lambda_j+Ck^3N^2/L^2 \quad \text{(B20)}$$

$$\|\mathbf{w}_d^n\|_0^2+kv\left[|\mathbf{w}_d^n|_1^2+\sum_{j=n_1}^{n_i}|\mathbf{w}_d^j|_1^2\right]+k\left[\|r_d^n\|_0^2\sum_{j=n_1}^{n_i}\|r_d^j\|_0^2\right]$$

$$\leqslant CkL\sum_{j=d+1}^{l}\lambda_j+Ck^3N^2/L^2+Ck^3\sum_{j=n_1}^{L}\|T_h^j-T_d^j\|_0^2 \quad \text{(B21)}$$

If $k=O(L^{-2})$, then by (B20) and (B21) we obtain

$$\|\tau_d^n\|_0+k^{1/2}\gamma_0^{-1/2}\left[|\tau_d^n|_1+\sum_{j=n_1}^{n_i}|\tau_d^j|_1\right]$$

$$\leqslant Ck^{1/2}\left[\sum_{j=n_1}^{n_i}|\mathbf{w}_d^j|_1^2\right]^{1/2}+C\left(k^{1/2}\sum_{j=d+1}^{l}\lambda_j\right)^{1/2}+Ck \quad \text{(B22)}$$

$$\|\mathbf{w}_d^n\|_0+(kv)^{1/2}\left[|\mathbf{w}_d^n|_1+\sum_{j=n_1}^{n_i}|\mathbf{w}_d^j|_1\right]+k\left[\|r_d^n\|_0^2+\sum_{j=1}^{n_i}\|r_d^j\|_0^2\right]$$

$$\leqslant C\left(k^{1/2}\sum_{j=d+1}^{l}\lambda_j\right)^{1/2}+Ck+Ck^{3/2}\left[\sum_{j=n_1}^{n_i}\|\tau_d^j\|_0^2\right]^{1/2} \quad \text{(B23)}$$

Combining (B22) and (B23), by Lemma 3.2, we obtain (29). □

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